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On Bellman's equations for mean and variance control of a Markov diffusion

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Abstract

A controlled diffusion process is considered with cost functions of "mean and variance" type. A regularization is proposed for computing the value of the cost function via Bellman's equations. The latter equations are in particular useful because they imply sufficiency of markovian strategies, at least, for regularized versions of processes. For the diffusion *without control*, a system of two well-posed linear PDEs is derived, similar to Kac's and Dynkin's moment equations, along with an equivalent single degenerate equation which turns out to be well-posed, too.

Keywords: degenerate Markov diffusion, mean and variance control, Bellman's equation
MSC: 60H10, 93E20.

1 Introduction

The idea of mean-variance optimization goes back to the Nobel Laureate in Economics Harry Markowitz [15]. During half a century it served as the basis for various mathematics-based studies in Economics and Finance. In the last two decades this concept was extensively investigated in modern stochastic financial

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theories. Föllmer and Sondermann [6] initiated the study of this approach for semimartingale models of stochastic markets, continued further in [5], [16], and in the following years in [16], [14], [7], et al. Unlike in the more classical theory of stochastic control, Bellman's equations, – often called HJB for Hamilton–Jacobi–Bellman – did not seem to play any significant role, except in the linear–quadratic (LQ) theory [19] which will be briefly commented later. The purpose of this paper is to consider a control problem for a general diffusion process, and for a non-quadratic cost function of mean and variance type, and to show how Bellman's equations may be useful in this setting. The cost functionals that we study are not most general. So, we do not include the "payoff at expiry" or "final payment" in order to avoid certain technical difficulties and to make the idea simple, leaving this more general case for further studies. Notice that in the classical control theory of diffusion processes (see [11]) the payoff at expiry does not bring any significant news. However, our situation is slightly different. What we propose is a *regularized* Bellman equation with an additional parameter $\psi \in R^1$, similar to that in [19], where regularization was not needed. We also show some natural bounds for the quality of regularization. Remark that is remains open whether it is possible to avoid regularization at all, like in LQ theory.

Let us consider a d-dimensional SDE driven a d-dimensional Wiener process $(W_t, \mathcal{F}_t,\, t \geq 0)$

$$dX_t = b(\alpha_t, t, X_t)\, dt + \sigma(\alpha_t, t, X_t)\, dW_t, \quad t \geq t_0, \qquad X_{t_0} = x. \tag{1}$$

For properties of the coefficients b and $sigma$, see section 3 below. The strategy $(\alpha_t,\, t_0 \leq t \leq T)$ may be chosen from the class $\mathcal{A}$ of all progressive measurable processes with values in $A \subset R^\ell$ such that $A \neq \emptyset$, bounded and closed. We will use the standard short notation where the dependence of X on the strategy, initial data x and t_0 is shown by $E^\alpha_{t_0,x}$ with respect to expectation; the full notation would be $X_t^{\alpha,t_0,x}$. It is well known that for the optimization problem

$$v^1(t,x) = \sup_{\alpha \in \mathcal{A}} E^\alpha_{t,x} \int_t^T f(\alpha_s, s, X_s)\, ds, \tag{2}$$

the following Bellman's equation should be considered, see [11],

$$\sup_{u \in A} \left(\left(\frac{\partial}{\partial t} + L^{(u)} \right) v^1 + f^{(u)} \right)(t,x) = 0, \qquad v(T,x) \equiv 0, \tag{3}$$

where

$$L^{(u)}(t,x) = \sum_{i,j} a_{ij}(u,t,x) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_j b_j(u,t,x) \frac{\partial}{\partial x^j},$$

where $a = \sigma\sigma^*$ is a $d \times d$ matrix. In the *mean and variance control* problem,

$$v(t,x) := \sup_{\alpha \in \mathcal{A}} \left\{ E^\alpha_{t,x} \int_t^T f(s, X_s)\, ds - \theta \ \mathrm{Var}^\alpha_{t,x} \left(\int_t^T f(s, X_s)\, ds \right) \right\}, \tag{4}$$

a customer wishes to maximize the expected profit, *simultaneously reducing the variance*. The risk aversion of the customer is expressed through the negative sign of the variance and the constant θ. The less θ, the less sensitive is the customer to a big variance, and vice versa. Now we can state the main goal of this paper as to propose *equations* which would allow us to compute the value (4), at least, approximately, using PDE techniques: informally speaking, we would like to apply the idea of Bellman's equations to this problem. The whole procedure is, of course, more involved than the single Bellman equation for the problem (2): hardly anybody may expect here anything comparable. Zhou and Li ([19] et al.) considered a linear–quadratic setting and successfully used HJB equations. The two theories, LQ and "classical control" have many similarities, however, they do differ at some points, such as integrability assumptions, less explicit forms of the answers ect. Indeed, in the general setting for the problem (4) there is no explicit solution equivalent to the LQ one, but, nevertheless, the answer given below (see Theorem 4) is not hopeless from the computational point of view, in particular, because of the dimension of the auxiliary parameter.

To present the idea, we will discuss an easier case where the cost function involves just the second moment,

$$v^2(t,x) = \sup_{\alpha\in\mathcal{A}} E^{\alpha}_{t,x}\left(\int_t^T f(\alpha_s,s,X_s)\,ds\right)^2. \tag{5}$$

The cost function with infimum may be considered similarly; we notice that either attitude – that is, with sup or inf – may make sense, due to various reasons. The expression on the right hand side of (5), may be readily transformed into the form of a first moment setting (see (13) below),

$$v^2(t,x,0) = 2\sup_{\alpha\in\mathcal{A}} E^{\alpha}_{t,x,0}\int_t^T f(\alpha_s,s,X_s)Y_s\,ds, \tag{6}$$

using the auxiliary process

$$dY_s = f(\alpha_s,s,X_s^{t,x,\alpha})\,ds, \quad s\ge t, \qquad Y_t = 0. \tag{7}$$

In (6), a new index zero in the expectation $E^{\alpha}_{t,x,0}$ stands for the initial value of the process Y, that is, for $Y_0 = 0$. As usual with Bellman's equations, it will be useful to allow a variable initial data y for this new component,

$$dY_s = f(\alpha_s,s,X_s^{t,x,\alpha})\,ds, \quad s\ge t, \qquad Y_t = y. \tag{8}$$

Now the function in (5) can be regarded as a cost function for the controlled extended Markov diffusion (X,Y),

$$v^2(t,x,y) = 2\sup_{\alpha\in\mathcal{A}} E^{\alpha}_{t,x,y}\int_t^T f(\alpha_s,s,X_s)Y_s\,ds,$$

at $y = 0$. The trouble is that this extended process (X, Y) is strongly degenerate and so is the corresponding PDE,

$$\sup_{u\in\mathcal{A}} \Big[v^2_t + \sum_i^d b_i(u,t,x)v^2_{x_i} + \frac{1}{2}\sum_{i,j=1}^d a_{ij}(u,t,x)v^2_{x_ix_j} + f(u,t,x)v^2_y + f(u,t,x)y\Big] = 0,$$
$$v^2(T,x,y) \equiv 0. \tag{9}$$

Here one might try two proposals: the first, to study this degenerate equation as it is; the second, to regularize the equation with the help of a small additional diffusion with respect to the second process Y (we will call its solution Y^ε) and study the regularized version of the problem. In this paper we consider only the latter case in the section 3. The result of the section 3 – Theorem 3 – states existence and uniqueness of a single regularized Bellman's equation along with an assessment of the quality of the proposed regularization. As an immediate consequence, Markov strategies are sufficient for the regularized problem.

For the "full" mean and variance control, the main result of the section 4 – Theorem 4 – suggests a parametric family of regularized Bellman's equations depending on some parameter $\psi \in R^1$; there the cost function in (4) turns out to be a maximum of solutions to those equations over $-\infty < \psi < +\infty$. Similar to the algorithm in [19] for the LQ linear–quadratic case, although less satisfactory compared to both LQ and classical control, this algorithm, however, only uses a one-dimensional auxiliary parameter ψ and in particular only, its values from some bounded interval should be taken into account (see below the Corollary 1). This apparently makes the algorithm feasible from the computational point of view. Reasonable modelling examples should be the next step of this work in the near future.

In the introductory section 2 of this paper we derive linear parabolic equations in suitable Sobolev function classes for the second moment of a general integral functional of a non-linear Markov diffusion, for which we did not succeed to find appropriate references. Similar equations hold true for any finite moment, too, however, we do not pursue this goal here. Let us mention some preceding results in this direction. For parabolic problems, Kac gave the equations for moments of a *functional of a Wiener process*, see [8], and also [1]. For elliptic problems, Dynkin ([2, Chapter 13, §4]) suggested a chain of equations for all moments, known in some references as *Dynkin's chain of equations*, – under an exponential moment condition which provides analyticity of the Laplace transformation of the functional under investigation; naturally, this is a functional of some homogeneous Markov diffusion. Fitzsimmons and Pitman [4] gave a review of Feynman-Kac and moment Kac formulae for general integral functionals of stopped Markov diffusions, in terms of distribution measures of the stopped processes; although never shown, corresponding PDEs are apparently expected. In [18], a finite chain of equations has been derived and explored under conditions that guarantee only a finite number of moments for some divergent type diffusions.

Let us formulate, what is new in this paper, or, at least, what is presented in a new form: (1) a parabolic differential equation system for the second moment of a general functional of a non-homogeneous Markov diffusion and an *equivalent* single degenerate parabolic equation for the same object; (2)a regularized Bellman's equation system depending on a one-dimensional parameter for a controlled nonlinear Markov diffusion with cost functions of *mean and variance type* for some nonlinear integral functionals, along with a certain rate of convergence. The latter gives a principled opportunity to use Bellman's equations and, hence, Markov strategies, at least, for the regularized process (X, Y^ε). At the same time, we use only well known results on existence and uniqueness for Bellman's equations.

2 Linear PDE system for the second moment

In the sequel, T is any fixed positive real value. All solutions of parabolic PDEs will be considered in the Sobolev classes $W^{1,2}_{p,loc}$ with one derivative with respect to t and two with respect to x in generalized $L_{p,loc}$ sense; here

$$L_{p,loc} = \{v(t,x):\ 0 \le t \le T,\ x \in R^d,\ \sup_{z\in R^d} \int_0^T \int_{|x-z|\le 1} |v(t,x)|^p \, dt\, dx < \infty\}. \quad (10)$$

Denote $\overline{W}^{1,2} = \bigcap_{p>1} W^{1,2}_{p,loc}$. For the functions of three variables, $v(t,x,y)$, $0 \le t \le T$, $x, y \in R^d$, we will use a similar Sobolev class $\overline{W}^{1,2,2} = \bigcap_{p>1} W^{1,2,2}_{p,loc}$, which does not require any new comments.

In order to simplify our presentation we always assume the function f to be bounded. This can be relaxed, which we comment in remarks below.

2.1 Linear PDE *system* for diffusion without control

Throughout this section, the process $X^{t_0,x}_t$, where $0 \le t_0 \le T$, is assumed to be a *non-controlled* diffusion

$$X^{t_0,x}_t = x + \int_{t_0}^t b(s, X^{t_0,x}_s)\, ds + \int_{t_0}^t \sigma(s, X^{t_0,x}_s)\, dW_s, \qquad t \ge t_0. \quad (11)$$

The Borel measurable bounded function f does not depend on a control either, $f(t,x)$, $0 \le t \le T$, $x \in R^d$. Here b is a Borel measurable vector function of (t,x) with values in R^d, σ is a $d \times d$-matrix Borel measurable function, both bounded, and $\sigma\sigma^*$ uniformly non-degenerate. A solution of (11) exists on some filtered probability space $(\Omega, P, \mathcal{F})$, $\quad \mathcal{F}_t, \quad t \ge 0$ with a $d-$dimensional Wiener process (see [10, Theorem 2.6.1]). Let us also assume that the function $a(t,\cdot) := \sigma\sigma^*(t,\cdot)$ is uniformly continuous, and that this continuity is uniform with respect to $0 \le t \le T$. Then, the solution of the equation (11) will be weakly unique ([17]), and, hence, for each (t_0, x) the process $(X^{t_0,x}_t,\ t \ge t_0)$ is strong Markov [9].

Consider the function

$$V(t_0,x) = E_{t_0,x}\Big(\int_{t_0}^T f(s,X_s)\,ds\Big)^2. \tag{12}$$

We have, due to the Fubini theorem,

$$\begin{aligned} V(t_0,x) &= E_{t_0,x}\Big(\int_{t_0}^T f(s,X_s)\,ds\Big)\Big(\int_{t_0}^T f(t,X_t)\,dt\Big) \\ &= 2E_{t_0,x}\bigg(\int_{t_0}^T f(t,X_t)\Big(\int_t^T f(s,X_s)\,ds\Big)\,dt\bigg) \\ &= 2E_{t_0,x}\bigg(\int_{t_0}^T f(t,X_t)E_{t_0,x}\Big(\int_t^T f(s,X_s)\,ds \mid \mathcal{F}_t\Big)\,dt\bigg) \\ &= 2E_{t_0,x}\bigg(\int_{t_0}^T f(t,X_t)\,U(t,X_t)\,dt\bigg), \end{aligned} \tag{13}$$

due to the Markov property, where

$$U(t,x) := E_{t,x}\int_t^T f(s,X_s)\,ds.$$

Proposition 1 *The function $U(t,x)$ is the unique solution in $\overline{W}^{1,2}$ of the equation,*

$$U_t + \sum_{i=1}^d b_i(t,x)U_{x_i} + \frac{1}{2}\sum_{i,j=1}^d a_{ij}(t,x)U_{x_ix_j} = -f(t,x), \tag{14}$$

with terminal condition:

$$U|_{t=T} = 0.$$

See [10, Chapter 4, §7].

Theorem 1 *There exists a function $V(t_0,x)$, which is the unique solution in $\overline{W}^{1,2}$ of the equation,*

$$V_t + \sum_{i=1}^d b_i(t,x)V_{x_i} + \frac{1}{2}\sum_{i,j=1}^d a_{ij}(t,x)V_{x_ix_j} = -2f(t,x)U(t,x), \tag{15}$$

with terminal condition $V(T,x)\equiv 0$.

Proof. Follows similarly from [10, Chapter 4, §7], due to the Proposition 1 and (13), since U is bounded.

2.2 A single linear PDE for diffusion without control

Remind the equation, equivalent to the equation (9) without control,

$$v^2_t + \sum_{i=1}^d b_i(t,x)v^2_{x_i} + \frac{1}{2}\sum_{i=1}^d\sum_{j=1}^d a_{ij}(t,x)v^2_{x_ix_j} + f(t,x)v^2_y + f(t,x)y = 0,$$

$$v^2(T,x,y) \equiv 0. \tag{16}$$

Simultaneously, consider the equation (14). We already know that there is a solution to the latter equation in the appropriate Sobolev class. Hence, we obtain the following result.

Theorem 2 *There is a unique solution of the equation (16) in the class of functions $v^2 : v^2(\cdot,\cdot,y) \in \overline{W}^{1,2}$ for each y, such that*
(1) the function v^2 is affine in y, that is, say,

$$v^2(t,x,y) = \tilde{v}^2(t,x) + y\hat{v}^2(t,x),$$

and $\hat{v}^2$ is a solution to the equation (14) in $\overline{W}^{1,2}$;
(2) the function $\tilde{v}^2$ is a solution to the the equation (15) with $U = \hat{v}^2$ in $\overline{W}^{1,2}$.

All the assertions follow easily from the previous subsection.

Remark 1 *The assumption on function f in this section may be relaxed to $f \in L_p \bigcap L_{2p}$ with any $p \ge d+1$. We find it probable that $f \in L_{p,loc} \bigcap L_{2p,loc}$ with any $p \ge d+1$, as defined above in (10), also suffices, and that further extensions are possible such as a certain growth of $L_{p,loc}$ and $L_{2p,loc}$ norms at infinity. However, we cannot include the full LQ case, because non-degeneracy of diffusion is important for this approach.*

3 Control of the regularized second moment

In this section we return to the process described by equation (1). We do not change notations, but use the same symbols σ, b, f now for functions of three variables, (u,t,x), $u \in A$, $0 \le t \le T$, $x \in R^d$. To simplify both the references and our presentation, in the sequel we assume the following conditions, which may be actually relaxed (cf. [11, Chapter 3]):

- The functions σ, b, f are Borel with respect to (u,t,x), continuous with respect to (u,x) and continuous with respect to x uniformly over u for each t; moreover,
- $\|\sigma(u,t,x) - \sigma(u,t,x')\| \le K|x-x'|$,

- $\|\sigma(u,t,x)\| + |b(u,t,x)\| + |f(u,t,x)| \le K$,
- $|f(u,t,x) - f(u,t,x')| \le K\,|x - x'|$.

In order to regularize the resulting parabolic equation and restore non-degeneracy for the process (8), we add to the equation another independent Wiener process with a small constant diffusion coefficient,

$$Y_t^{y,\varepsilon} = Y_t^{t_0,x,y,\alpha,\varepsilon} = y + \int_0^t f(\alpha_t, t, X_t^{t_0,x,\alpha})\,dt + \varepsilon \tilde{W}_t. \tag{17}$$

Correspondingly, consider the optimization problem

$$v^{2,\epsilon}(t,x,y) = \sup_{\alpha\in\mathcal{A}} E_{t,x} \int_t^T f(\alpha_s, s, X_s) Y_t^{y,\epsilon}\,dt. \tag{18}$$

We will see that for small ε, this function $v^{2,\varepsilon}$ is close to the function v^2,

$$v^2(t,x,y) = \sup_{\alpha\in\mathcal{A}} E_{t,x} \int_t^T f(\alpha_s, s, X_s) Y_t^{y}\,dt, \tag{19}$$

Although we restrict ourselves to the case of bounded functions f, more general cases could be considered. Notice that the PDE for the ε-case is a (nonlinear) non-degenerate parabolic PDE, for which we can easily prove existence and uniqueness of solutions, and which can be solved numerically,

$$\begin{aligned} &\sup_u \Big[v_t^{2,\varepsilon} + \sum_{i=1}^d b_i(u,t,x) v_{x_i}^{2,\varepsilon} + \frac{1}{2}\sum_{i,j=1}^d a_{ij}(u,t,x) v_{x_i x_j}^{2,\varepsilon} \\ &+ f(u,t,x) v_y^{2,\varepsilon} + \frac{\varepsilon^2}{2} v_{yy}^{2,\varepsilon} + f(u,t,x) y \Big] = 0, \qquad v^{2,\varepsilon}\Big|_{t=T} = 0. \end{aligned} \tag{20}$$

Theorem 3 *The equation (20)has a unique solution $v^{2,\varepsilon}$ of in the class $\overline{W}^{1,2,2}$. Moreover,*

$$\sup_{t,x,y} \left| \left(v^{2,\varepsilon} - v^2\right)(t,x,y) \right| \le C\varepsilon.$$

Proof. The fact that there is a unique solution of (20) follows, e.g., from [11] similarly to the Proposition 1. To show the second assertion we estimate,

$$\begin{aligned} & v^{2,\varepsilon} - v^2 \\ = \;& \sup_{\alpha\in\mathcal{A}} E^{\alpha}_{t,x,y} \int_t^T f(\alpha_s, s, X_s) Y_s^{\varepsilon}\,ds - \sup_{\alpha\in\mathcal{A}} E^{\alpha}_{t,x,y} \int_t^T f(\alpha_s, s, X_s) Y_s\,ds \\ \le \;& \sup_{\alpha\in\mathcal{A}} \left(E^{\alpha}_{t,x,y} \int_t^T f(\alpha_s, s, X_s) Y_s^{\varepsilon}\,ds - E^{\alpha}_{t,x,y} \int_t^T f(\alpha_s, s, X_s) Y_s\,ds \right) \\ = \;& \sup_{\alpha\in\mathcal{A}} E^{\alpha}_{t,x,y} \int_t^T f(\alpha_s, s, X_s)(Y_s^{\varepsilon} - Y_s)\,ds \end{aligned}$$

$$
\begin{aligned}
&= \sup_{\alpha\in\mathcal{A}} E^{\alpha}_{t,x,y} \int_t^T f(\alpha_s, s, X_s)\varepsilon(\tilde W_s - \tilde W_t)\, ds \\
&\le \|f\|_B \sup_{t\le s\le T} E|\tilde W_s - \tilde W_t|(T-t)\,\varepsilon = \|f\|_B (T-t)^{\frac{3}{2}} E|\tilde W_1|\,\varepsilon \\
&= \|f\|_B (T-t)^{\frac{3}{2}} \sqrt{\frac{2}{\pi}}\,\varepsilon. \qquad (21)
\end{aligned}
$$

Similarly we obtain a lower bound,

$$
v^{2,\varepsilon} - v^2 \ge -\|f\|_B (T-t)^{\frac{3}{2}} \sqrt{\frac{2}{\pi}}\,\varepsilon
$$

and the Theorem 3 is proved.

Remark 2 *Similarly to the previous section, here the assumption on function* f *may be relaxed to* $\sup_u |f(u,\cdot)| \in L_p \bigcap L_{2p}$ *with any* $p \ge d+1$*. The bounds like (21) should use those* L_p *and* L_{2p} *norms instead of* $\|f\|_B$*. Again it is probable that* $\sup_u |f(u,\cdot)| \in L_{p,loc} \bigcap L_{2p,loc}$ *with any* $p \ge d+1$ *also suffices, and that further extensions are possible related to a certain growth of* $L_{p,loc}$ *and* $L_{2p,loc}$ *norms at infinity. The full LQ case is not included.*

4 Mean-Variance Control

The next and final goal of the paper is to maximize a linear combination of the mean and variance of a payoff function, and suggest some suitable equations. Along with the function v^1, – see (2) above, – consider the following function,

$$
v^{\varepsilon}(t,x,y) := \sup_{\alpha\in\mathcal{A}} \left\{ v^{1,\alpha}(t,x) - \theta v^{2,\varepsilon,\alpha}(t,x,y) + \theta\,(v^{1,\alpha}(t,x))^2 \right\}, \qquad (22)
$$

where standard notations have been used,

$$
v^{1,\alpha}(t,x) := E^{\alpha}_{t,x} \int_t^T f(\alpha_s, s, X_s)\, ds, \qquad (23)
$$

$$
v^{2,\varepsilon,\alpha}(t,x) := E^{\alpha}_{t,x,y} \int_t^T f(\alpha_s, s, X_s) Y^{\varepsilon}_s\, ds. \qquad (24)
$$

Later we will also need the notation

$$
v^{2,\alpha}(t,x) := E^{\alpha}_{t,x,y} \int_t^T f(\alpha_s, s, X_s) Y_s\, ds. \qquad (25)
$$

The key point in the representation is that the term $(v^{1,\alpha})^2(t,x)$ in (22) may be represented in the form,

$$
(v^1)^2 = \sup_{\psi} \{-\psi^2 - 2\psi v^1\}. \qquad (26)
$$

Then the optimization problem (22) can be rewritten as

$$\begin{aligned} v^{\varepsilon}(t,x,y) &= \sup_{\alpha\in\mathcal{A}}\sup_{\psi}\left[v^{1,\alpha}(t,x)+\theta[-\psi^2-2\psi v^{1,\alpha}(t,x)]-\theta v^{2,\varepsilon,\alpha}(t,x,y)\right] \\ &= \sup_{\psi}\left[\sup_{\alpha\in\mathcal{A}}\left(v^{1,\alpha}(t,x)[1-2\theta\psi]-\theta v^{2,\varepsilon,\alpha}(t,x,y)\right)-\theta\psi^2\right], \qquad (27) \end{aligned}$$

where we will be finally interested in $y=0$. Given ψ, denote

$$V^{1,\varepsilon}(t,x,y,\psi) := \sup_{\alpha\in\mathcal{A}}\left(v^{1,\alpha}(t,x)[1-2\theta\psi]-\theta v^{2,\varepsilon,\alpha}(t,x,y)\right), \qquad (28)$$

$$V^{1}(t,x,y,\psi) := \sup_{\alpha\in\mathcal{A}}\left(v^{1,\alpha}(t,x)[1-2\theta\psi]-\theta v^{2,\alpha}(t,x,y)\right), \qquad (29)$$

and let

$$L^{(u,\varepsilon)} := \sum_{i,j} a_{ij}(u,t,x)\frac{\partial^2}{\partial x^i\partial x^j}+\sum_j b_j(u,t,x)\frac{\partial}{\partial x^j}+\frac{\varepsilon^2}{2}\Delta_{yy}+f(u,t,x)\frac{\partial}{\partial y}.$$

The function in the left hand side of (28) is a solution of the Bellman PDE,

$$\begin{aligned} &\sup_u\left[\left(\frac{\partial}{\partial t}+L^{(u,\varepsilon)}\right)V^{1,\varepsilon}(t,x,y,\psi)+[1-2\theta\psi+\theta y]f(x)\right]=0, \\ &V^{1,\varepsilon}\Big|_{t=T}=0. \end{aligned} \qquad (30)$$

Under our standing assumptions above, we have existence and uniqueness of solutions of the equation (30), due to [10, Chapters 3 and 4]. Now, the external optimization problem in (27) takes the form

$$v^{\varepsilon}(t,x,y)=\sup_{\psi}\left[V^{1,\varepsilon}(t,x,y,\psi)-\theta\psi^2\right]. \qquad (31)$$

Of course, this is not just a simple quadratic equation with respect to ψ, since the function $V^{1,\varepsilon}$ depends on ψ. However, the function $V^{1,\varepsilon}$, being locally Lipschitz in ψ, grows in this variable at most linearly, as shown in the next Lemma below. Hence, the supremum is attained at some ψ from a certain bounded interval.

Lemma 1 *The functions $V^1(t,x,y,\psi)$ and $V^{1,\varepsilon}(t,x,y,\psi)$ satisfy the bounds,*

$$|V^{1,\varepsilon}(t,x,y,\psi)|+|V^1(t,x,y,\psi)|\le C\,(1+|\psi|),$$

and

$$\max\left(|V^1(t,x,y,\psi)-V^1(t,x,y,\psi')|,|V^{1,\varepsilon}(t,x,y,\psi)-V^{1,\varepsilon}(t,x,y,\psi')|\right)\le C\,|\psi-\psi'|.$$

Proof. The linear growth follows from the definitions (28) and (29), since all three functions $v^{2,\varepsilon}(\alpha,\ldots)$, $v^2(\alpha,\ldots)$ and $v^1(\alpha,\ldots)$ are bounded uniformly in *alpha*.

Let us show that the function $V^{1,\varepsilon}$ is locally Lipschitz with respect to ψ; the function V^1 may be considered similarly. We estimate,

$$
\begin{aligned}
&V^{1,\varepsilon}(t,x,y,\psi) - V^{1,\varepsilon}(t,x,y,\psi') \\
= \;& \sup_{\alpha\in\mathcal{A}}\Big(v^{1,\alpha}(t,x)[1-2\theta\psi] - \theta v^{2,\varepsilon,\alpha}(t,x,y)\Big) \\
- \;& \sup_{\alpha\in\mathcal{A}}\Big(v^{1,\alpha}(t,x)[1-2\theta\psi'] - \theta v^{2,\varepsilon,\alpha}(t,x,y)\Big) \\
\le \;& 2\theta\sup_{\alpha\in\mathcal{A}}\Big|v^{1,\alpha}(t,x)\Big|\,|\psi'-\psi| \le C\,\theta\,|\psi'-\psi|.
\end{aligned}
$$

The correspnding lower bound follows similarly, which proves the Lemma 1.

Corollary 1 *Any supremum in (31) is attained at $|\psi|\le C$, where C is a constant from the second inequality in Lemma 1.*

Naturally, *some* supremum *is* attained.

Theorem 4 *The approximate cost function v^ε satisfies (31), and there exists a constant C such that*

$$
\sup_{t,x,y}|v^\varepsilon(t,x,y) - v(t,x,y)| \le C\,\theta\,\varepsilon. \tag{32}
$$

Proof. Only (32) needs to be established. We have, due to the calculus in (21),

$$
\begin{aligned}
&v^\varepsilon(t,x,y) - v(t,x,y) \\
&\le \sup_{\psi}\Big(V^{1,\varepsilon}(t,x,y,\psi) - \theta\psi^2 - V^1(t,x,y,\psi) + \theta\psi^2\Big) \\
&= \sup_{\psi}\Bigg(\sup_{\alpha\in\mathcal{A}}\Big(v^{1,\alpha}(t,x)[1-2\theta\psi] - \theta v^{2,\varepsilon,\alpha}(t,x,y)\Big) \\
&\qquad - \sup_{\alpha\in\mathcal{A}}\Big(v^{1,\alpha}(t,x)[1-2\theta\psi] - \theta v^{2,\alpha}(t,x,y)\Big)\Bigg) \\
&\le \sup_{\psi}\sup_{\alpha\in\mathcal{A}}\Big(v^{1,\alpha}(t,x)[1-2\theta\psi] - v^{1,\alpha}(t,x)[1-2\theta\psi] \\
&\qquad +\theta\Big(v^{2,\alpha}(t,x,y) - v^{2,\varepsilon,\alpha}(t,x,y)\Big)\Big) \\
&= \theta\sup_{\alpha\in\mathcal{A}}\Big(v^{2,\alpha}(t,x,y) - v^{2,\varepsilon,\alpha}(t,x,y)\Big) \le \theta\,C\,\varepsilon.
\end{aligned}
$$

The lower bound follows similarly. Hence, Theorem 4 is proved.

Remark 3 *Similarly to the previous sections, here the assumption on function f may be relaxed to $\sup_u |f(u,\cdot)| \in L_p \bigcap L_{2p}$ with any $p \geq d+1$, and, possibly, further to a certain local version. This will change some bounds in the calculus, and we postpone the detailed presentation till a further paper.*

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