MATH 1035 ANALYSIS: 2011

Summary of results

For the examination on this module, you should know the **statements** of all definitions, facts, and theorems given in the lectures and in the hand-outs, and the **proofs** of all facts and theorems, **unless** stated otherwise below. You will be expected to apply the results to specific examples; these examples are likely to be similar to examples discussed in class or set as homework questions.

The format of the 3-hour examination is that there will be 6 questions - 2 on the lectures of Dr. Daws and 4 on my lectures. You should attempt 4 questions in total. All questions will have the same maximum mark. (The examination counts for 85% of the marks and the homeworks count for 15%.)

All answers should be given clearly and accurately in correct English sentences; give explicit and precise definitions, etc. General phrases, such as 'a_n gets close to ℓ as n gets big', that are not mathematically accurate will receive no credit; statements that do not make sense will be penalized, so think carefully before you write doubtful material. If necessary, concentrate on learning some theorems and proofs very well, and writing them in a clear and beautiful way, rather than making a slap-dash attempt at too many questions.

Warning Some statements given below are very brief; see the lecture notes, or the book of Hart, for a fuller version.

Chapter 1 : Ordered fields

The idea of a totally ordered field (see handout); you do not need the details of this definition. The ordered fields $\mathbb{Q}$ and $\mathbb{R}$. Notation for intervals such as $[a,b]$ and (a,b).

Fact Take $a,b \in \mathbb{R}$ with $a < b$. Then there is a rational x with $a < x < b$ and an irrational y with $a < y < b$.

Chapter 2 : The completeness axiom

Definition of an *upper bound*, the *supremum*, and the *maximum* of a non-empty subset of an ordered field.

Note that, for a set S which is bounded above, we have $x_0 = \sup S$ if and only if $s \leq x_0$ for all $s \in S$ and $(x_0 - \varepsilon, x_0] \cap S \neq \emptyset$ for each $\varepsilon > 0$.

Definition *An ordered field is* complete *if every non-empty subset which is bounded above has a supremum.*

Remark : $\mathbb{Q}$ is not complete; for example, the set $\{x \in \mathbb{Q}^+ : x^2 < 2\}$ is bounded above, but it has no supremum in $\mathbb{Q}$.

Definition *The* real number system $\mathbb{R}$ *is a totally ordered field that is complete.* This is the **completeness axiom** for $\mathbb{R}$.

Theorem (Archimedean property) *For each $a, b \in \mathbb{R}$ with $a, b > 0$, there exists $n \in \mathbb{N}$ with $na > b$.*

Fact For each $c \in \mathbb{R}^+$ and each $n \in \mathbb{N}$, there exists a unique $x \in \mathbb{R}^+$ with $x^n = c$.

Chapter 3 : Sequences

Definition *A* sequence *(in* $\mathbb{R}$*) is a map from* $\mathbb{N}$ *to* $\mathbb{R}$*.*

We denote a sequence by (x_n) or $(x_n : n \in \mathbb{N})$. Distinguish between a sequence and its range, which is the set $\{x_n : n \in \mathbb{N}\}$.

Definition *A sequence* (x_n) tends to infinity *means: given* $M \in \mathbb{R}$*, there exists* $n_0 \in \mathbb{N}$ *such that* $x_n \geq M$ *for each* $n \geq n_0$*.*

Write either '$x_n \to \infty$ as $n \to \infty$' or '$\lim\limits_{n\to\infty} x_n = \infty$'.

Definition *A sequence* (x_n) converges *to* $\ell \in \mathbb{R}$ *means: given* $\varepsilon > 0$*, there exists* $n_0 \in \mathbb{N}$ *such that* $|x_n - \ell| < \varepsilon$ *for each* $n \geq n_0$*.*

Write either '$x_n \to \ell$ as $n \to \infty$' or '$\lim\limits_{n\to\infty} x_n = \ell$'.

Definition *A sequence* (x_n) *is:*

(i) increasing *if* $x_{n+1} \geq x_n$ *for each* $n \in \mathbb{N}$*;*

(ii) bounded above *if there exists* $M \in \mathbb{R}$ *such that* $x_n \leq M$ *for each* $n \in \mathbb{N}$*;*

(iii) bounded *if both* (x_n) *and* $(-x_n)$ *are bounded above.*

Facts (i) An increasing sequence which is bounded above converges to the supremum of its range.

(ii) A convergent sequence is bounded.

Examples (i) Suppose that $\alpha > 0$. Then $n^\alpha \to \infty$ as $n \to \infty$.

(ii) Suppose that $|r| < 1$. Then $r^n \to 0$ as $n \to \infty$.

(iii) Suppose that $\alpha > 0$ and $r > 1$. Then $n^\alpha / r^n \to 0$ as $n \to \infty$.

(iv) The sequence $((1 + 1/n)^n : n \in \mathbb{N})$ is increasing, and $\lim_{n\to\infty} (1 + 1/n)^n = \mathrm{e}$. [This is the definition of e.]

(v) $n^{1/n} \to 1$ as $n \to \infty$.

(vi) Suppose that $a > 0$. Then $a^{1/n} \to 1$ as $n \to \infty$.

Facts Let $x_n \to a$ and $y_n \to b$ as $n \to \infty$. Then: (i) $x_n + y_n \to a + b$; (ii) $x_n y_n \to ab$; (iii) $x_n / y_n \to a/b$ (provided that $b \neq 0$); (iv) $|x_n| \to |a|$.

Fact Suppose that $x_n > 0$ and $x_n \to 0$. Then $1/x_n \to \infty$.

Theorem (Squeeze rule) Let (x_n), (y_n), and (z_n) be three sequences. Suppose that there exists $n_0 \in \mathbb{N}$ such that $x_n \leq y_n \leq z_n$ for each $n \geq n_0$ and that $x_n \to \ell$ and $z_n \to \ell$. Then $y_n \to \ell$.

Definition *Let* (x_n) *be a sequence. A* subsequence *of* (x_n) *has the form* $(x_{n_k} : k \in \mathbb{N})$*, where* $n_{k+1} > n_k$ *for each* $k \in \mathbb{N}$*.*

Fact A sequence that is not convergent may have many convergent subsequences.

Fact Let (x_n) be a sequence that is not bounded above. Then (x_n) has a subsequence $(x_{n_k} : k \in \mathbb{N})$ such that $\lim_{k\to\infty} x_{n_k} = \infty$.

Chapter 4 : Series

Definition *Let* (a_k) *be a sequence in* $\mathbb{R}$*. The* n^{th} partial sum *of* (a_k) *is* $s_n = \sum_{k=1}^{n} a_k$*. The series* $\sum_{k=1}^{\infty} a_k$ converges to s*, or* has sum s*, if the sequence* (s_n) *of partial sums converges to* s*. A series* diverges *if it does not converge.*

Here we often write $\sum_{k=1}^{\infty} a_k$ or just $\sum a_k$ for the series (and for its sum, if it converges).

Examples (i) **Geometric series** Suppose that $|r| < 1$. Then the series $\sum_{k=0}^{\infty} r^k$ converges and has sum $1/(1-r)$. Suppose that $|r| \geq 1$. Then the series does not converge.

(ii) **Harmonic series**. The series $\sum_{k=1}^{\infty} 1/k$ diverges.

Fact Suppose that $\sum a_k = a$ and $\sum b_k = b$. Then $\sum(\alpha a_k + \beta b_k) = \alpha a + \beta b$ for $\alpha, \beta \in \mathbb{R}$.

Fact Suppose that $\sum a_k$ converges. Then $a_k \to 0$ as $k \to \infty$. But this necessary condition for convergence is not sufficient for convergence; see the harmonic series.

Fact (Comparison test) (i) Suppose that $a_k \geq 0$ and $b_k \geq 0$ for all $k \in \mathbb{N}$. Then $a_k = O(b_k)$ if there exists $k_0 \in \mathbb{N}$ and $m > 0$ such that $a_k \leq m b_k$ for each $k \geq k_0$. Suppose that $\sum b_k$ converges and $a_k = O(b_k)$. Then $\sum a_k$ converges.

(ii) We write $a_k \sim b_k$ if $a_k = O(b_k)$ and $b_k = O(a_k)$: this is the case if $a_k/b_k \to \ell$ for some $\ell \neq 0$. Suppose that $a_k \sim b_k$. Then $\sum a_k$ converges if and only if $\sum b_k$ converges.

Facts (Ratio test) Let (a_k) be a sequence with $a_k > 0$ for each $k \in \mathbb{N}$, and suppose that $a_{k+1}/a_k \to \ell$ as $k \to \infty$.

(i) If $\ell < 1$, then $\sum a_k$ converges.
(ii) If $\ell > 1$, then $\sum a_k$ diverges.
(iii) If $\ell = 1$, there is no information.

Facts (Root test) Let (a_k) be a sequence with $a_k > 0$ for each $k \in \mathbb{N}$, and suppose that $a_k^{1/k} \to \ell$ as $k \to \infty$.

(i) If $\ell < 1$, then $\sum a_k$ converges.
(ii) If $\ell > 1$, then $\sum a_k$ diverges.
(iii) If $\ell = 1$, there is no information.

You will be expected to apply appropriate tests to determine whether or not various series converge or diverge; the series will be similar to those in the notes or the homeworks.

Definition *A series* $\sum a_k$ converges absolutely *if* $\sum |a_k|$ *converges.*

Fact An absolutely convergent series is convergent.

Definition *Let* f *be a function on* $[a, \infty)$ *for some* $a \in \mathbb{R}$*. Then* $f(x) \to \ell$ *as* $x \to \infty$ *means: given* $\varepsilon > 0$*, there exists* $x_0 \geq a$ *such that* $|f(x) - \ell| < \varepsilon$ *for each* $x \geq x_0$*.*

An improper integral $\int_a^{\infty} f(t)\mathrm{d}\,t$ converges to ℓ *means that* $\int_a^x f(t)\mathrm{d}\,t \to \ell$ *as* $x \to \infty$*.*

Example $\int_{k=1]\infty t^{-\alpha}\mathrm{d}\,t}$ converges if and only if $\alpha > 1$.

Theorem (Integral test) *Let* $f : [1, \infty) \to \mathbb{R}^+$ *be a continuous, decreasing function. Then* $\sum_{k=1}^{\infty} f(k)$ *converges if and only if* $\int_1^{\infty} f(t)\mathrm{d}\,t$ *converges.*

Example $\sum_1^{\infty} 1/k^{\alpha}$ converges if and only if $\alpha > 1$.

Definition *An* alternating series *has the form* $\sum_{k=0}^{\infty} (-1)^k a_k$*, where* $a_k \geq 0$*.*

Theorem (Alternating series test) *Suppose that* $a_k \geq 0$ *for each* $k \in \mathbb{N}$*, that* (a_k) *is decreasing, and that* $a_k \to 0$ *as* $k \to \infty$*. Then the alternating series* $\sum_{k=0}^{\infty}(-1)^k a_k$ *converges.*

Example For each $\alpha > 0$, the series $\sum_{k=1}^{\infty}(-1)^{k+1}/k^{\alpha}$ converges, but it converges absolutely only if $\alpha > 1$.

Definition A *power series* has the form $\sum_{k=0}^{\infty} a_k x^k$, where (a_k) is a fixed sequence.

Theorem *For each such power series* $\sum_{k=0}^{\infty} a_k x^k$*, there exists* $R \in [0, \infty]$ *such that:*

(i) *if* $0 < R < \infty$*, then the series converges absolutely for* $|x| < R$ *and diverges for* $|x| > R$ *(with no information for* $|x| = R$*);*

(ii) *if* $R = 0$*, then the series diverges for* $x \neq 0$*;*

(iii) *if* $R = \infty$*, then the series converges absolutely for all* $x \in \mathbb{R}$*.*

The 'number' R is called the *radius of convergence* of the power series. Suppose that $|a_{k+1}| / |a_k| \to \ell$ as $k \to \infty$ (this does not always happen). Then $R = 1/\ell$ (suitably interpreted).

Chapter 5 : Continuous functions

Recall : if $f : S \to T$ is a function, then S is the *domain* of f and T is the *codomain* of f.

Definition *Let* $a \in \mathbb{R}$*. A* neighbourhood *of* a *is an open interval in* $\mathbb{R}$ *of the form* $(a - \delta, a + \delta)$*, where* $\delta > 0$*. A* deleted neighbourhood *has the form* $\{x \in \mathbb{R} : 0 < |x - a| < \delta\}$*, where* $\delta > 0$*.*

Definition *Suppose that* f *is defined on a deleted neighbourhood of* a*, so that* f *is defined* near a*. Then* $\lim_{x \to a} f(x) = \ell$ *means: given* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that* $|f(x) - \ell| < \varepsilon$ *whenever* $0 < |x - a| < \delta$*. The function* f *is* continuous at a *if* f *is defined on a neighbourhood of* a *and* $\lim_{x \to a} f(x) = f(a)$*.*

There are similar definitions of one-sided and infinite limits, such as $\lim_{x \to a+} f(x) = \ell$, of $\lim_{x \to a} f(x) = \infty$, $\lim_{x \to \infty} f(x) = \ell$, and of $\lim_{x \to \infty} f(x) = \infty$.

The following important theorem allows us to reduce proofs about sums and products of continuous functions to analogous results about sequences.

Theorem *The following are equivalent statements:*

(a) f *is continuous at* a*;* (b) $f(x_n) \to f(a)$ *whenever* $x_n \to a$ *in* $\mathbb{R}$*.*

Similar statements apply to other forms of the limit operation.

Facts Suppose that $\lim_{x \to a} f(x) = \ell$ and $\lim_{x \to a} g(x) = m$. Then:

(i) $\lim_{x \to a}(f + g)(x) = \ell + m$; (ii) $\lim_{x \to a}(fg)(x) = \ell m$; (iii) $\lim_{x \to a}(f/g)(x) = \ell/m$ (provided that $m \neq 0$); (iv) $\lim_{x \to a} |f(x)| = |\ell|$.

Theorem (Squeeze rule) *Suppose that* $f(x) \leq h(x) \leq g(x)$ *on a deleted neighbourhood of* a*, and that* $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \ell$*. Then* $\lim_{x \to a} h(x) = \ell$*.*

Definition *Suppose that* f *is defined on a subset* S *of* $\mathbb{R}$*, and that* $s_0 \in S$*. Then* f *is* continuous at s_0 *if, given* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that* $|f(s) - f(s_0)| < \varepsilon$ *whenever* $0 < |s - s_0| < \delta$ *and* $s \in S$*. The function* f *is* continuous on S *if it is continuous at each point of* S*.*

Write $C(S)$ for the set of continuous functions on S; it is a vector space, which means that we have $\alpha f + \beta g \in C(S)$ whenever $\alpha, \beta \in \mathbb{R}$ and $f, g \in C(S)$. Also $f \cdot g \in C(S)$ whenever $f, g \in C(S)$. Write $C[a, b]$ for $C([a, b])$, etc.

Examples (i) all polynomials are continuous on $\mathbb{R}$;
(ii) the rational function p/q is continuous on $\{x \in \mathbb{R} : q(x) \neq 0\}$.

Fact (Glue rule) Suppose that f is continuous on $[a, b]$, that g is continuous on $[b, c]$, and that $f(b) = g(b)$. Define $h(x) = f(x)$ for $a \leq x \leq b$ and $h(x) = g(x)$ for $b \leq x \leq c$. Then h is continuous on $[a, c]$.

Theorem (Composition rule) *Suppose that f is continuous at a and that g is continuous at $f(a)$. Then $g \circ f$ is continuous at a.*

Similarly, if $f \in C(S)$ and $g \in C(f(S))$, then $g \circ f \in C(S)$.
The trigonometric functions sin and cos (defined geometrically) are continuous on $\mathbb{R}$, and

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1\,.$$

Set $\tan \theta = \sin \theta / \cos \theta$. Then tan is continuous on $\mathbb{R} \setminus \{n\pi + \pi/2 : n \in \mathbb{Z}\}$.

Facts Take $a \in \mathbb{R}$. Suppose that f is defined near a and continuous at a.
(i) The function f is bounded on some neighbourhood of a.
(ii) Suppose that $r < f(x) < s$ for x in a deleted neighbourhood of a. Then $r \leq f(a) \leq s$.
(iii) Suppose that $r < f(a) < s$. Then there exists $\delta > 0$ such that $r < f(x) < s$ whenever $|x - a| < \delta$.

Theorem *Let $f \in C[a, b]$. Then f is bounded above (i.e., there exists $M \in \mathbb{R}$ such that $f(x) \leq M$ whenever $a \leq x \leq b$), and f attains its (upper) bound (i.e., there exists $c \in [a, b]$ such that $f(x) \leq f(c)$ whenever $a \leq x \leq b$).*

Theorem (Intermediate value theorem – IVT) *Let $f \in C[a, b]$. Suppose that $f(a) < 0$ and $f(b) > 0$. Then there exists $c \in (a, b)$ with $f(c) = 0$.*

Applications to finding solutions of equations. For example, each polynomial of odd degree has a root in $\mathbb{R}$. The following is a consequence of IVT and the above theorem.

Theorem (Interval theorem) *Let $f \in C[a, b]$ with $m = \inf f([a, b])$ and $M = \sup f([a, b])$. Then the range of f is the closed interval $[m, M]$ in $\mathbb{R}$.*

Theorem (Inverse function theorem) *Let $f \in C[a, b]$. Suppose that $f(a) < f(b)$ and that f is injective. Set $c = f(a)$ and $d = f(b)$. Then the range of f is $[c, d]$, and $f^{-1} : [c, d] \to [a, b]$ is continuous.*

The statement and proof of the inverse function theorem is **not** required.

Chapter 6 : Differentiation

Definition *Let f be defined on a neighbourhood of $a \in \mathbb{R}$. Then f is* differentiable at *a if*

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{1}{h}(f(a + h) - f(a))$$

exists; if so, the limit is denoted by $f'(a)$.

Example Let $f(x) = |x|$. This is continuous at $x = 0$, but not differentiable at $x = 0$.

Definition *Let f be defined on (a, b). Then f is* differentiable on (a, b) *if it is differentiable at c for each $c \in (a, b)$. The function $f' : c \mapsto f'(c)$ is the* derived function*, or* first derivative *of f.*

Theorem *Suppose that f is differentiable at a. Then f is continuous at a.*

Theorem *Suppose that f and g are differentiable at a. Then so are $\alpha f + \beta g$, fg, and f/g (provided that $g(a) \neq 0$). Further:*

$$\begin{aligned}(\alpha f + \beta g)'(a) &= \alpha f'(a) + \beta g'(a)\,;\\ (fg)'(a) &= f'(a)g(a) + f(a)g'(a)\,;\\ (f/g)'(a) &= (f'(a)g(a) - f(a)g'(a))/g(a)^2\,.\end{aligned}$$

Examples (i) Let $n \in \mathbb{N}$. Then Z^n has derived function nZ^{n-1}. Here $Z^n(t) = t^n$ for $t \in \mathbb{R}$.

(ii) $\sin' = \cos$, $\cos' = \sin$, etc.

Theorem (Chain rule) *Suppose that f is differentiable at a and that g is differentiable at $f(a)$. Then $g \circ f$ is differentiable at a, and $(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$.*

Theorem (Inverse function theorem) *Suppose that f is continuous and strictly increasing on $[a, b]$, and suppose further that f is differentiable at $c \in (a, b)$. Then the inverse function f^{-1} is differentiable at $d = f(c)$, and its derivative is*

$$(f^{-1})'(d) = \frac{1}{f'(c)} = \frac{1}{f'(f^{-1}(d))}\,.$$

The proof of this inverse function theorem is **not** required.

Theorem *Let f be continuous on $[a, b]$ and differentiable on (a, b), so that $f \in D[a, b]$.*

(i) (Rolle) *Suppose that $f(a) = f(b)$. Then there exists $c \in (a, b)$ with $f'(c) = 0$.*

(ii) (Mean value theorem - MVT) *There exists $c \in (a, b)$ with*

$$f'(c) = \frac{f(b) - f(a)}{b - a}\,.$$

Suppose that f' exists and is continuous on $[a, b]$. It follows from the mean value theorem that $|f(b) - f(a)| \leq M\,|b - a|$, where $M = \sup\{|f'(c)| : a < c < b\}$.

Theorem *Let $f \in D[a, b]$.*

(i) *Suppose that $f'(c) = 0$ whenever $c \in (a, b)$. Then f is constant.*

(ii) *Suppose that $f'(c) > 0$ whenever $c \in (a, b)$. Then f is strictly increasing on (a, b).*

Theorem (Cauchy's MVT) *Let $f, g \in D[a, b]$. Then there exists $c \in (a, b)$ with*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}\,.$$

Theorem (l'Hopital's rule) *Let f and g be differentiable on a deleted neighbourhood of $a \in \mathbb{R}$. Suppose that:*

(i) $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = 0$ (ii) $g' \neq 0$ *on a neighbourhood of a, and* $\lim_{x\to a} f'(x)/g'(x) = L$.

Then $\lim_{x\to a} f(x)/g(x) = L$.

Applications to various examples.

HGD March 2011