MATH1035: Workbook Four

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Roots of unity

An important result which can be proved by induction is:

De Moivre's theorem (natural number case): Let $\theta \in \mathbb{R}$ and $n \in \mathbb{N}$. Then

$$\big(\cos(\theta) + i\sin(\theta)\big)^n = \cos(n\theta) + i\sin(n\theta).$$

Proof: The result is obviously true when $n = 1$. We proceed by induction, so suppose the claim is true for n. Then

$$\begin{aligned}
&\big(\cos(\theta) + i\sin(\theta)\big)^{n+1} \\
&\qquad= \big(\cos(\theta) + i\sin(\theta)\big)\big(\cos(n\theta) + i\sin(n\theta)\big) \quad \text{by the induction hypothesis,} \\
&\qquad= \big(\cos(\theta)\cos(n\theta) - \sin(\theta)\sin(n\theta)\big) + i\big(\cos(\theta)\sin(n\theta) + \sin(\theta)\cos(n\theta)\big) \\
&\qquad= \cos\big((n+1)\theta\big) + i\sin\big((n+1)\theta\big).
\end{aligned}$$

So the result is true by mathematical induction. □

Using the complex exponential notation, De Moivre's theorem says simply that

$$\big(e^{i\theta}\big)^n = e^{in\theta} \quad \text{for } n \in \mathbb{N}, \theta \in \mathbb{R}.$$

This looks *obvious*! However, it's not quite: we have to do a little work. Remember that we have previously proved that $e^{i\theta}e^{i\varphi} = e^{i(\theta+\varphi)}$ (indeed, the proof was rather similar to the above, using trigonometric double angle formulae: this is not a coincidence!) So, if $(e^{i\theta})^n = e^{in\theta}$ (which is true if $n = 1$) then

$$\big(e^{i\theta}\big)^{n+1} = e^{in\theta}e^{i\theta} = e^{i(n+1)\theta}.$$

So by induction, the exponential form of De Moivre's theorem is true (this was so simple, you *almost* didn't need to use induction).

We can prove a better result:

De Moivre's theorem: Let $\theta \in \mathbb{R}$ and $n \in \mathbb{Z}$. Then

$$\big(\cos(\theta) + i\sin(\theta)\big)^n = \cos(n\theta) + i\sin(n\theta).$$

Proof: When $n \geq 1$ we have already proved this. When $n = 0$, then as $z^0 = 1$ for any non-zero[1] $z \in \mathbb{C}$, and $\cos(0) + i\sin(0) = 1$, we have that the result holds when $n = 0$. When $n < 0$, let $m = -n \in \mathbb{N}$, so that

$$\begin{aligned}
\big(\cos(\theta) + i\sin(\theta)\big)^n &= \big(\cos(\theta) + i\sin(\theta)\big)^{-m} = \frac{1}{\big(\cos(\theta) + i\sin(\theta)\big)^m} \\
&= \frac{1}{\cos(m\theta) + i\sin(m\theta)} \qquad \text{by De Moivre for } m, \\
&= \frac{\cos(m\theta) - i\sin(m\theta)}{\cos^2(m\theta) + \sin^2(m\theta)} = \cos(m\theta) - i\sin(m\theta) \\
&= \cos(n\theta) + i\sin(n\theta),
\end{aligned}$$

which completes the proof. □

[1] Why is this true? To be really precise, this is actually a *definition*.

Again, this is very easy in the exponential form, we have already shown that for any $\theta \in \mathbb{R}$, we have $1/e^{i\theta} = e^{-i\theta}$. I'll leave it as an exercise to writeup the above proof using exponential notation.

De Moivre's theorem allows us to solve equations of the form $z^n = 1$, where $n \in \mathbb{Z}$. Notice that for $\theta \in \mathbb{R}$, if $n\theta = 2k\pi$ for some $k \in \mathbb{Z}$, then

$$\big(\cos(\theta) + i\sin(\theta)\big)^n = \cos(n\theta) + i\sin(n\theta) = \cos(2k\pi) + i\sin(2k\pi) = 1.$$

Thus, when $n \neq 0$, some solutions to $z^n = 1$ are

$$z = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right) \qquad \text{for } k = 0, 1, \cdots, n-1.$$

Notice that there is no point considering $k = n$, for then $2k\pi/n = 2\pi$ and then $\cos(2\pi) = \cos(0)$ and $\sin(2\pi) = \sin(0)$, so we just get the same answer as for $k = 0$. Similarly, if we try any other $k \in \mathbb{Z}$, we'll get the same result as for some[2] number in the set $\{0, 1, \cdots, n-1\}$.

Claim: Let $n \in \mathbb{N}$, $n \neq 0$. Then the solutions to $z^n = 1$ are

$$z = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right) \qquad \text{for } k = 0, 1, \cdots, n-1.$$

Proof: We showed that these are solutions, so we need to show that they are the *only* solutions. Let $z^n = 1$. Thus $|z^n| = |z|^n = 1$ (why? Again, technically, this is a proof by induction, but it's so simple I'll skip it). So $|z| = 1$, as $|z|$ is a positive real, and the only solution to $t^n = 1$ for $t \in \mathbb{R}$ with $t > 0$, is $t = 1$.

We showed before that if $|z| = 1$, then $z = \cos(\theta) + i\sin(\theta)$ for some $\theta \in \mathbb{R}$. Indeed, we can choose θ in the range $[0, 2\pi)$. So by De Moivre, $1 = z^n = \cos(n\theta) + i\sin(n\theta)$. So $\sin(n\theta) = 0$, which means that $n\theta$ is an integer multiple of π. Similarly, $\cos(n\theta) = 1$, which means that $n\theta$ is an integer multiple of 2π. So $n\theta = 2k\pi$ for some $k \in \mathbb{Z}$, that is, $\theta = 2k\pi/n$. As $0 \leq \theta < 2\pi$, we have

$$0 \leq \frac{2k\pi}{n} < 2\pi \implies 0 \leq 2k\pi < 2n\pi \implies 0 \leq k < n,$$

using that $n > 0$. So k is in the set $\{0, 1, \cdots, n-1\}$ as required. □

If $n < 0$, then $z^n = 1$ if and only if $1 = 1/z^n = z^{-n}$, and then we can apply the above theorem for $-n$.

We call solutions to $z^n = 1$ the *nth roots of unity*. This is a slightly old-fashioned phrase: "unity" means 1, and then we take the nth root of 1.

Question: What are the complex solutions to $z^2 = 1 + i$?

Answer: Write z in polar form as $z = r(\cos(\theta) + i\sin(\theta))$, so by De Moivre, $z^2 = r^2(\cos(2\theta) + i\sin(2\theta))$. Hence $z^2 = 1 + i$ is equivalent to $r^2 = |1 + i|$ and 2θ being the argument of $1 + i$. Now,

$$|1 + i| = \sqrt{1+1} = \sqrt{2} \Leftrightarrow r = 2^{1/4}.$$

The Principal Argument of $1 + i$ is $\pi/4$, and so the argument of $1 + i$ is $\pi/4$ or $2\pi + \pi/4$, and so forth. Hence $\theta = \pi/8$ or $\pi + \pi/8$ (you can check that any other choice would just be one of these added to a multiple of 2π). So

$$z^2 = 1 + i \Leftrightarrow z = 2^{1/4}\big(\cos(\pi/8) + i\sin(\pi/8)\big) \text{ or } z = 2^{1/4}\big(\cos(\pi + \pi/8) + i\sin(\pi + \pi/8)\big).$$

Of course, the second solution is just the negative of the first, as we were solving a quadratic equation in this specific case. □

[2] You might like to think: which number?

Quantifiers: $\forall$ and $\exists$

You will have seen phrases like "there exists" or "for all" quite a bit by now. Just as we write $\Rightarrow$ to mean "implies", we write $\forall$ to mean "for all": remember this as "upside down A for all". We write $\exists$ to mean "there exists": remember this as "backwards E for there exists". These are called *quantifiers*.

We can now translate phrases in English into symbols. An example:

$$\big(\forall\, x \in \mathbb{R}\big)[x^2 \geq 0] \quad \text{means} \quad \text{"For all real numbers } x\text{, we have that } x^2 \geq 0.\text{"}$$

We use the brackets simply to make the result easier to read. Just translating, you might have come up with "For all $x \in \mathbb{R}$, $x^2 \geq 0$", but this is still hard to read, so we translate a bit more to the version I have. (Notice that this statement is true).

Some further examples:

- $\big(\exists\, a \in \mathbb{R}\big)[a^2 = 2]$ which means "There is a real number a, with $a^2 = 2$". We can make this even clearer: "The square-root of 2 is a real number".
- $\big(\forall\, x \in \mathbb{C}\big)\big(\forall\, y \in \mathbb{C}\big)[xy = yx]$ means: "For all complex numbers x and y, we have $xy = yx$". We can say this is a simpler way: "Multiplication of complex numbers is commutative".
- $\big(\forall\, \theta \in \mathbb{R}\big)\big(\forall\, n \in \mathbb{Z}\big)[(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)]$. This is just De Moivre's theorem!

Things get a bit more complicated when we combine both $\forall$ and $\exists$. For example,

$$\big(\forall\, r \in \mathbb{R}\big)\big(\forall\, s \in \mathbb{R}\big)\big(\exists\, t \in \mathbb{C}\big)[rs = t^2].$$

This (true) statement means that for all $r, s \in \mathbb{R}$, there exists $t \in \mathbb{C}$ with $t^2 = rs$. You could also say: "For any two real numbers $r, s \in \mathbb{R}$, there is a square-root of rs in $\mathbb{C}$".

Fix a function $f : X \to Y$. Then consider:

$$\big(\forall\, y \in Y\big)\big(\exists\, x \in X\big)[f(x) = y]$$

This means: "For each $y \in Y$ there is some $x \in X$ with $f(x) = y$". This is the definition of what it means for f to be *surjective*! Notice that this statement could be true or false, depending upon what f is.

Consider now

$$\big(\exists\, y \in \mathbb{Z}\big)\big(\forall\, x \in \mathbb{Z}\big)[x < y].$$

So we translate: "There exists an integer y, for all integers x, we have that x is less than y". If we think about this for a little while, we see that this is equivalent to "There exists an integer y such that all integers are less than y". This is false of course: as $y + 1$ is not less than y. Let us think about this a bit further: we cannot find a *single* counter-example: instead, we have to find a new counter-example for each y. But this is easy, for given any $y \in \mathbb{Z}$, we can find $x \in \mathbb{Z}$ with $x \not< y$; just let $x = y + 1$ (or $x = y$ or $x = y + 100$ or whatever).

If we swap the order around, and consider

$$\big(\forall\, x \in \mathbb{Z}\big)\big(\exists\, y \in \mathbb{Z}\big)[x < y],$$

then we do get a true statement. If we translate, we get "For all integers x, there exists an integer y with y bigger than x". This is true, because if I give you x, then you can certainly find some y with $x < y$, for example, $y = x + 1$ would work. Notice how I phrased this as a "game" or "challenge": I think this is a good way to think about such problems.

Again, let $f : X \to Y$ be a function, and consider:

$$\big(\forall\, x \in X\big)\big(\forall\, y \in X\big)[x \neq y \implies f(x) \neq f(y)].$$

This means: "For any choice of x and y in X, if $x \neq y$ then $f(x) \neq f(y)$." This is the definition of what it means for f to be *injective*: "different elements of X get mapped by f to different elements of Y". We saw before that this is equivalent to:

$$\big(\forall\, x \in X\big)\big(\forall\, y \in X\big)[f(x) = f(y) \implies x = y].$$

This 2nd condition is easier to check. Notice that I used the "implies" symbol, but when I translated, I used the "if ... then ... " form, which I think is easier to understand.

We can go the other way, and convert mathematics statements which are in English into statements in symbols. We can also work with general sets. Some examples:

- Let $A = \{1, 2, 3, 4, 5, 6\}$. Then "there is a natural number greater than all elements of A" can be translated to $(\exists\, x \in \mathbb{N})(\forall\, y \in A)[x > y]$.
- We can translate "Every natural number is either even or odd" as $(\forall\, x \in \mathbb{N})(\exists\, n \in \mathbb{N})[x = 2n \text{ or } x = 2n - 1]$. How did I do this? Well, a natural number is even if it's of the form $2n$ for some $n \in \mathbb{N}$; a natural is odd if it's of the form $2n - 1$.
- Let $B = (1, 123) \subseteq \mathbb{R}$. Then "the square of any number x in B is greater than x" is $(\forall\, x \in B)[x^2 > x]$.
- Finally, an example of a false statement. "There is a greatest real number smaller than 1". Let's define a set $C = (-\infty, 1)$, so C is the collection of all real numbers less than 1. Then our statement is that C has a greatest element. This can now be written as $(\exists\, x \in C)(\forall\, y \in C)[x \geq y]$.

  Finally, *why* is this false?

Equivalence relations

We want to abstract the idea of "equality", which will ultimately lead us to a rigourous definition of the rational numbers.

A *relation* on a set X is a very general idea, which we introduce by giving some examples:

1. Numbers x and y are related if and only if $x = y$.
2. An integer x is related to an integer y if and only if x divides y.
3. An person p is related to a person q if and only if p is a brother or sister of q.
4. Complex numbers w and z are related if and only if $|w - z| < 1$.

We denote a relation by R, and for $x, y \in X$, we write xRy if x is related to y. For example, we can re-write the above as:

1. On $\mathbb{R}$ (or $\mathbb{C}$ or ...) xRy if and only if $x = y$.
2. On $\mathbb{Z}$, define xRy if and only if x divides y.
3. On the set of people, define pRq if and only if p is a brother or sister of q.
4. On $\mathbb{C}$, define wRz if and only if $|w - z| < 1$.

There are some common properties of relations. Let R be a relation on X. Then:

- R is *reflexive* if xRx for all $x \in X$.
- R is *symmetric* if xRy implies that yRx.
- R is *transitive* if, whenever xRy and yRz, then xRz as well.

We can write these using quantifiers:

- R is *reflexive* if $(\forall\, x \in X)[xRx]$.
- R is *symmetric* if $(\forall\, x \in X)(\forall\, y \in X)[xRy \implies yRx]$.
- R is *transitive* if $(\forall\, x \in X)(\forall\, y \in X)(\forall\, z \in X)[xRy \text{ and } yRz \implies xRz]$.

Let us consider our examples:

1. $xRy \Leftrightarrow x = y$.

   This is reflexive, as $xRx \Leftrightarrow x = x$ which is always true.

   This is symmetric, as if xRy then $x = y$ so also $y = x$ so yRx.

   This is transitive, as if xRy and yRz, then $x = y$ and $y = z$, so also $x = y = z$ so $x = z$ so xRz.

2. On $\mathbb{Z}$, $xRy \Leftrightarrow x$ divides y.

   This is reflexive, because for any $x \in \mathbb{R}$, certainly x divides x, so xRx holds.

   This is not symmetric: a *counter-example* is that $5R10$, because 5 divides 10, but $10 \not R 5$, because 10 does not divide 5. (As usual, a single counter-example is enough!)

   This is transitive, for if xRy and yRz, then x divides y and y divides z. That means I can find $k, l \in \mathbb{Z}$ with $y/x = k$ and $z/y = l$. But then $z/x = ly/x = lk$ which is an integer, showing that x divides z, that is, xRz.

3. pRq if and only if p is a brother or sister of q.

   This is not reflexive, because a person is not a sibling of themselves!

   This is symmetric: if pRq then p is a sister (or brother) of q, so that q is a sister (or brother) of p, so qRp.

   This is also transitive, for if pRq and qRs, then p and q are siblings, and also q and s are siblings, so p and s are also siblings, so pRs.

4. On $\mathbb{C}$, define wRz if and only if $|w - z| < 1$.

   This is reflexive, as for any $w \in \mathbb{C}$, we have that $|w - w| = 0 < 1$, so wRw.

   This is also clearly symmetric: if wRz then $|w - z| < 1$ so also $|z - w| < 1$ so zRw.

   It is not transitive: a counter-example is found by setting $x = 0$, $y = 4i/5$ and $z = 4i/5 - 4/5$. Then $|x - y| = |0 - 4i/5| = 4/5$ and $|y - z| = |4i/5 - 4i/5 + 4/5| = 4/5$ so xRy and yRz. However, $|x - z| = |0 - 4i/5 + 4/5| = \sqrt{32/25} > 1$ so $x \not R z$.

Definition: An *equivalence relation* is a relation which is reflexive, symmetric and transitive. We tend to write $\sim$ instead of R for an equivalence relation.

Let's do a long worked example.

Claim: On $\mathbb{C}$, $z \sim w$ if and only if $e^z = e^w$ defines an equivalence relation.

Proof: This is reflexive, as $e^z = e^z$ for any z, so $z \sim z$. This is symmetric, for $z \sim w \implies e^z = e^w \implies e^w = e^z \implies w \sim z$. Similarly, this is transitive, for if $z \sim w$ and $w \sim x$ then $e^z = e^w$ and $e^w = e^x$ so $e^z = e^x$, so $z \sim x$. □

Suppose we fix $z = a + bi \in \mathbb{C}$. Which $w \in \mathbb{C}$ satisfy $z \sim w$? Well, let $w = c + di$ so

$$z \sim w \;\Leftrightarrow\; e^z = e^w \;\Leftrightarrow\; e^a e^{bi} = e^c e^{di} \;\Leftrightarrow\; e^a = e^c \text{ and } e^{bi} = e^{di}.$$

Taking the logarithm, we see that $e^a = e^c$ if and only if $a = c$. However,

$$e^{bi} = e^{di} \;\Leftrightarrow\; e^{di-bi} = e^0 = 1 \;\Leftrightarrow\; \cos(d-b) = 1 \text{ and } \sin(d-b) = 0.$$

This is equivalent to $d - b$ being an integer multiple of 2π. That is, there exists $k \in \mathbb{Z}$ with $d - b = 2\pi k$. We have shown:

Claim: For $z = a + bi \in \mathbb{C}$ fixed, we have that $w = c + di$ is such that $z \sim w$ if and only if $c = a$ and $d = b + 2\pi k$ for some $k \in \mathbb{Z}$.

However, if $c = a$ and $d = b + 2\pi k$ then $w = c + di = a + bi + 2\pi ki = z + 2\pi ki$. So really we have:

Claim: For $z \in \mathbb{C}$ fixed, we have that $z \sim w$ if and only if $w = z + 2\pi ki$ for some $k \in \mathbb{Z}$. Or, in symbols, $z \sim w \;\Leftrightarrow\; (\exists\, k \in \mathbb{Z})[w = z + 2\pi ki]$.

Ideas for the tutorial

Here are some ideas for things which might be interesting to discuss; "answers" will be put on the VLE.

➡It might be a good idea to draw the solutions to $z^5 = 1$ on the Argand diagram. Do similar, or maybe slightly more complicated, examples.

➡A typical question using De Moivre is: Express $(\sin\theta)^3$ using $\sin(\theta)$ and $\sin(3\theta)$. *Hint:* We have two ways of working out $(\cos\theta + i\sin\theta)^3$, namely, De Moivre, and just expanding out. Then look at the imaginary parts.

➡Think about the final example in the "quantifiers" section: namely, that "There is a great real number smaller than 1" is false.

➡Here is a "proof" that a relation R is reflexive if it's both symmetric and transitive. Indeed, suppose xRy, so as R is symmetric, we have that yRx. Thus xRy and yRx, so as transitive, also xRx. Thus R is reflexive.

Why is this not true? (Think: find a counter-example!) What, *precisely* was my mistake?

➡You probably know that a quadratic equation has at most 2 *roots* (or solutions). A cubic has a most three roots. A polynomial of degree n (meaning the highest power involved is n) has a most n roots. If we believe this, then the argument on the middle of page 2 is made a lot easier! However, why does a degree n polynomial have at most n roots?

Problem Set 4

Due in at the lecture on Wednesday 17 November.

In the following, I'll often write "short proof". You should now be starting to do some work in rough, and then be writing up neat final answers to hand in. By "short proof", I mean a nicely written, short and to-the-point proof: similar to the way I wrote proofs out above. It's a learning process to decide how much detail you need to give in a proof: ask in tutorials if you are unsure if your proofs are too short or too long.

1. For the complex number questions, there are electronic resources on the VLE which will help you.

   (a) Write the following, in the simplest way possible, in the form $a + bi$.

   i. $\big(\cos(3\pi/28) + i\sin(3\pi/28)\big)^7$

   ii. $\big(\cos(2\pi/3) - i\sin(2\pi/3)\big)^{11}$

   iii. $(\sqrt{3} + i)^{201}$. *Hint:* Convert to the form $r(\cos(\theta) + i\sin(\theta))$.

   (b) Find the solutions, with $z \in \mathbb{C}$, to the following equations (see the example at the end of page 2).

   i. $z^5 = 1$

   ii. $z^8 = -1$

   iii. $z^6 - (2 + i) = 0$. *Hint: Convert* $2 + i$ *into polar form.*

2. (a) Convert the following into words. Try, if possible, to find the "nicest" form, not just "For all ..., there exists ...".

   In each case, decide if the result is true (and if so, write a *short* proof: some things are so obvious they don't need a proof!) or false (and if so, give a counter-example).

   i. $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})[x^2 = y]$.

   ii. $(\forall y \in \mathbb{R})(\exists x \in \mathbb{R})[x^2 = y]$.

   iii. $(\forall y \in \mathbb{N})(\exists x \in \mathbb{N})[x^2 = y]$.

   iv. $(\exists x \in \mathbb{R})[x^2 + 3x + 2 = 0]$.

   v. $(\exists x \in \mathbb{N})[x^2 + 3x + 2 = 0]$.

   vi. $(\forall x \in \mathbb{N})(\exists y \in \mathbb{C})[y\overline{y} = x]$.

   (b) Convert the following into symbols.

   i. There exist two prime numbers whose sum is even. You might want to define a set, $P = \{\text{prime numbers}\}$.

   ii. For each natural number n, there is a prime number bigger than n.

   iii. There exists a rational number greater than $\sqrt{5}$.

   iv. For any real numbers a and b, there exists a solution in $\mathbb{C}$ to the equation $x^2 + ax + b = 0$.

   (c) Let $A = \{1, 2, 3\}$ and $B = [0, \infty)$. For each of the following, prove that the statement is true, or give a counter-example. (You don't have to translate the symbols into words, but obviously this might help you!)

   i. $(\forall x \in A)(\exists y \in \mathbb{Q})[y > x]$.

   ii. $(\forall x \in B)(\exists y \in \mathbb{Q})[y > x]$.

   iii. $(\exists y \in \mathbb{Z})(\forall x \in A)[y > x]$.

   iv. $(\exists y \in \mathbb{Z})(\forall x \in B)[y > x]$.

   v. $(\exists x \in A)[x^2 \leq x]$.

3. For the following relations, decide if they are reflexive (or not), symmetric (or not), and transitive (or not). You do *not* need to give a proof, but do give a *counter-example* if you think the property does not hold.

(a) On $\mathbb{R}$, let xRy if and only if $x - y \in \mathbb{N}$.

(b) On $\mathbb{R}$, let xRy if and only if $(\exists\, n \in \mathbb{Z})[y = 10^n \times x]$.

(c) On $\mathbb{Q} \times \mathbb{Q}$, let $(x_1, x_2)R(y_1, y_2)$ if and only if $x_1 = y_2$.

4. The following are equivalence relations. In each case, give a short proof that we do have an equivalence relation. Fix some x, and then find a simple condition on y which tells us when $x \sim y$.

   For example, above we showed that $z \sim w \Leftrightarrow e^z = e^w$ is an equivalence relation. Then $z \sim w$ is equivalent to $w = z + 2\pi k i$ for some $k \in \mathbb{Z}$.

   (a) On $\mathbb{C}$, define $x \sim y$ if and only if $\mathrm{Re}(x) = \mathrm{Re}(y)$.

   (b) On $\mathbb{Z}$, define $x \sim y$ if and only if 10 divides $x - y$. Again, it might help to re-write this as $x \sim y \Leftrightarrow (\exists\, k \in \mathbb{Z})[x - y = 10k]$.

   (c) Let $Q = \{(x_1, x_2) \in \mathbb{Z} \times \mathbb{Z} : x_2 \neq 0\}$ (in words, Q is the set of ordered pairs in $\mathbb{Z} \times \mathbb{Z}$ with a non-zero second co-ordinate). On Q, define $(x_1, x_2) \sim (y_1, y_2)$ if and only if $x_1 y_2 = x_2 y_1$.

Optional questions for further practise

1. Find the solutions, with $z \in \mathbb{C}$, to the following equations.

   (a) $z^2 + 2z + 4 = 0$

   (b) $z^8 + 2z^4 + 4 = 0$

2. Convert the following into words. Are they true or false?

   (a) $(\forall\, x \in \mathbb{R})(\exists\, y \in \mathbb{R})[x + y = 1]$.

   (b) $(\exists\, y \in \mathbb{R})(\forall\, x \in \mathbb{R})[x + y = 1]$.

   (c) $(\forall\, x \in \mathbb{N})(\exists\, y \in \mathbb{N})[x + y = 1]$.

3. The following are equivalence relations. Prove this; then fix x and find a "simple" description of those y with $x \sim y$.

   (a) Let X and Y be sets, and $f : X \to Y$ a function. For $x, y \in X$ define $x \sim y$ if and only if $f(x) = f(y)$.

   (b) On $\mathbb{C}$, define $x \sim y$ if and only if $|x| = |y|$.

   (c) For $x \in \mathbb{R}$, let $\lfloor x \rfloor$ be the biggest integer less than or equal to x. So $\lfloor 1 \rfloor = 1$, $\lfloor \pi \rfloor = 3$, $\lfloor \sqrt{17} \rfloor = 4$ and $\lfloor -0.6 \rfloor = -1$. On $\mathbb{R}$, we define $x \sim y$ if and only if $\lfloor x \rfloor = \lfloor y \rfloor$.
   Hint: It might help to prove that for any $x \in \mathbb{R}$, we have $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$.