MATH1035: Workbook Three

–M. Daws, 2010

More on functions

Remember that for a function $f : X \to Y$, we say that X is the *domain*, and that Y is the *codomain* or *target*. The sets X and Y are part of the definition of f.

Definition: Let $f : X \to Y$ be a function. The *image* of f is the subset of Y given by $\{f(x) : x \in X\}$. That is, the image is the subset of Y which we get by mapping X under f. We also say *range* for the *image*.

For example, let $f : \mathbb{N} \to \mathbb{Z}$ be given by $f(n) = n + 10$, and let $g : \mathbb{N} \to \mathbb{Z}$ be given by $g(n) = n - 2$. Then both f and g have domain $\mathbb{N}$ and codomain $\mathbb{Z}$. However, the image of f is

$$\{f(n) : n \in \mathbb{N}\} = \{n + 10 : n \in \mathbb{N}\} = \{11, 12, 13, \cdots\} = \{n \in \mathbb{Z} : n \geq 11\},$$

while the image of g is

$$\{g(n) : n \in \mathbb{N}\} = \{n - 2 : n \in \mathbb{N}\} = \{-1, 0, 1, 2, \cdots\} = \{n \in \mathbb{Z} : n \geq -1\}.$$

So the images of f and g are different (and, in both cases, the image **does not** equal the codomain).

Definition: Let $f : X \to Y$ be a function. We say that f is *surjective* if the image of f is equal to the codomain of f. We might also say that f is a *surjection*, or that f *surjects*, or that f is an *onto function*, or simply that f is *onto*.

So a function $f : X \to Y$ is surjective if, given any element $y \in Y$, we can find some $x \in X$ with $f(x) = y$. "We can get everything in Y by applying f to something in X".

So neither of the examples above surject (neither of the above example functions are onto). Consider the function

$$h : \mathbb{Z} \to \mathbb{Z}, \text{ given by } h(n) = n + 1.$$

The codomain of this is $\mathbb{Z}$. This does surject. Why? Because if $m \in \mathbb{Z}$ is any integer, then we can find some $n \in \mathbb{Z}$ (the domain of h) with $h(n) = m$. Indeed, just let $n = m - 1$.

So this is a mini-proof: we have to pick something arbitrary in the codomain, and prove that there exists something in the domain which maps to it under our function. To show that a function isn't surjective, we have to find something in the codomain which is not mapped to. This is rather like finding a counter-example.

Definition: Let $f : X \to Y$ (I can miss out adding "is a function", as this notation tells us that f is a function). Then f is *injective* if "different things get mapped to different things". In symbols, this means that for each $a, b \in X$, if $a \neq b$ then also $f(a) \neq f(b)$. This is equivalent to the statement that, for $a, b \in X$, if $f(a) = f(b)$, then $a = b$ (and this is usually easier to check). We might also say that f is an *injection*, or that f *injects*, or that f is a *one-to-one function*, or simply that f is *one-to-one*.

Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = e^x$. Is this injective? Well, let $s, t \in \mathbb{R}$ and suppose that $f(s) = f(t)$. So $e^s = e^t$. Taking the logarithm,[1] we find that $s = \log(e^s) = \log(e^t) = t$, so $s = t$. As s and t were arbitrary, this shows that f is injective. (Notice that f is not surjective!) Again, this is a mini-proof.

Let $g : \mathbb{Z} \to \mathbb{N}$ be given by $g(n) = n^2 + 2$. This is not injective, as $g(-2) = 6 = g(2)$, so two different elements of the domain get mapped to the same element of the codomain. Again, this is very like finding a counter-example.

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[1] I *always* work to base e, so the logarithm means the natural logarithm: maybe you would usually write this as ln and not log.

Definition: Let $f : X \to Y$. Then f is *bijective* if it is both injective and surjective. We say that f is a *bijection*, or that f *bijects*.

It will be useful now to introduce a little more notation, which uses the symbol ∞. **This is not a number** but should be thought of as purely notation, meaning something like "continue forever".

Definition: Let $x \in \mathbb{R}$. We define

$$(x, \infty) = \{t \in \mathbb{R} : x < t\}, \qquad [x, \infty) = \{t \in \mathbb{R} : x \leq t\},$$
$$(-\infty, x) = \{t \in \mathbb{R} : t < x\}, \qquad (-\infty, x] = \{t \in \mathbb{R} : t \leq x\}.$$

The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = e^x$ is not bijective, as it is not surjective. However, the function $f : \mathbb{R} \to (0, \infty)$; $f(x) = e^x$ is a bijection. It is injective for the same reason as above, but now it is also surjective. Indeed, given any $y \in (0, \infty)$, let $x = \log(y)$, so that $f(x) = e^x = y$, that is, f maps to all of $(0, \infty)$. So f is a bijection.

Claim: Let $f : X \to Y$ be a bijective function. There is a (unique) function $g : Y \to X$ such that $g(f(x)) = x$ and $f(g(y)) = y$ for each $x \in X$ and $y \in Y$. We call g the *inverse* to f, and write f^{-1} for g.

For example, given the function $f : \mathbb{R} \to (0, \infty)$ defined by $f(x) = e^x$, we have that $f^{-1} : (0, \infty) \to \mathbb{R}$ is the function $f^{-1}(x) = \log(x)$. Notice that $f^{-1}(x)$ **is not** $1/f(x)$.

Definition: Let $f : X \to Y$ and $g : Y \to Z$. The *composition* of f and g is $g \circ f : X \to Z$, and is the function we get by applying f first, and then applying g, so $(g \circ f)(x) = g(f(x))$ for any $x \in X$. (I write $(g \circ f)(x)$ and not $g \circ f(x)$ to avoid confusion).

For a set X, the *identity function* $\mathrm{id} : X \to X$ is the function defined by $\mathrm{id}(x) = x$.

So the inverse function f^{-1} satisfies $f \circ f^{-1} = \mathrm{id}$ and $f^{-1} \circ f = \mathrm{id}$.

Cartesian products and the Argand Diagram

Definition: Let X and Y be sets. An *ordered pair*, in X and Y, is (x, y) where $x \in X$ and $y \in Y$. We say that x is the *first coordinate* and that y is the *second coordinate*. The *Cartesian Product* of X and Y is denoted $X \times Y$, and is the set of all ordered pairs. So $X \times Y = \{(x, y) : x \in X, y \in Y\}$. We often write X^2 for $X \times X$, and $X^3 = X \times X \times X$, and so forth.

You probably already know that $\mathbb{R}^2$ is the plane, and that $\mathbb{R}^3$ represents three-dimensional space. In $\mathbb{R}^2$, we have that $(1, 2)$ and $(2, 1)$ are not the same. This is why we say "*ordered* pair". [2]

Suppose that $X = \{a, b\}$ and $Y = \{1, 2, 3\}$. Then $X \times Y$ is the set

$$X \times Y = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}.$$

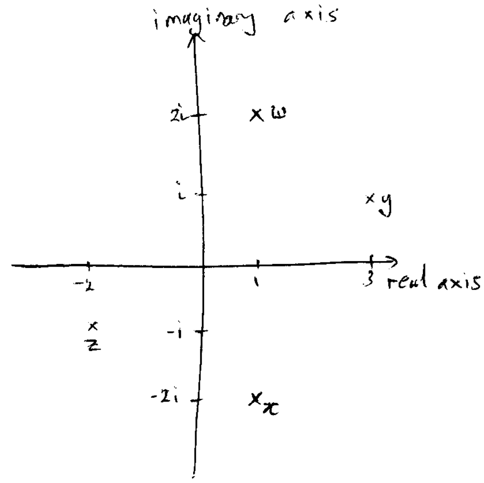
We have seen that any complex number can be written as $a + bi$ for some $a, b \in \mathbb{R}$. This allows us to identify $\mathbb{C}$ with $\mathbb{R}^2$. A complex number $z = a + bi$ is identified with the ordered pair (a, b). Or, $z \in \mathbb{C}$ is identified with $(\mathrm{Re}(z), \mathrm{Im}(z))$. Formally, we might define a map

$$\theta : \mathbb{C} \to \mathbb{R}^2 \text{ given by } \theta(z) = (\mathrm{Re}(z), \mathrm{Im}(z)).$$

We can then easily check that θ is a bijection; so each element of $\mathbb{C}$ corresponds to a unique element of $\mathbb{R}^2$.

[2] Notice the clash of notation: $(1, 2)$ could also represent the set $\{t \in \mathbb{R} : 1 < t < 2\}$. It's usually clear from context what is meant.

We can visualise $\mathbb{R}^2$ as the plane. So now we can also visualise $\mathbb{C}$ as a plane. This is called the *Argand diagram.* Indeed, we often call the x-axis the *real axis* and the y-axis the *imaginary axis.*



A sketch of $w = 1 + 2i$, $x = 1 - 2i$, $y = 3 + i$ and $z = -1 - 2i$.
Notice that we often label the imaginary axis with i, $2i$, $-i$ and so forth (instead of $1, 2, -1$ etc.)

Claim: We can write any $z \in \mathbb{C}$ as $z = r(\cos(\theta) + i\sin(\theta))$ for some $r \geq 0$ and $\theta \in \mathbb{R}$. Here $r = |z|$ the modulus of z, and θ is the *argument of* z.

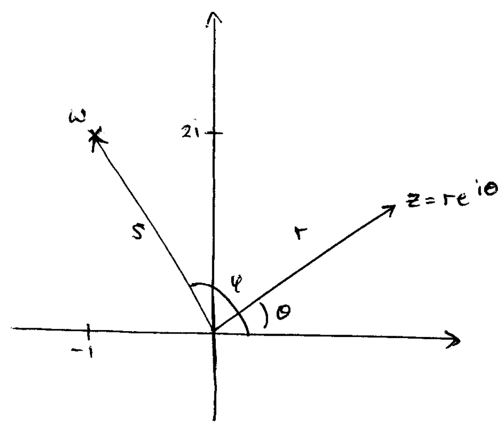
We saw the proof in lectures. Notice that we have lots of choices for θ, because $\cos(\theta + 2\pi) = \cos(\theta)$ and $\sin(\theta + 2\pi) = \sin(\theta)$. Indeed, we can add any integer multiple of 2π to θ and not change anything. It is usual to fix some range which θ must be in, which will ensure that θ is then unique. This gives the *principal argument*[3] which I have chosen to be $(-\pi, \pi]$. We write $\text{Arg}(z)$ for the principle argument of z, so by definition, $-\pi < \text{Arg}(z) \leq \pi$. It is also common to use $[0, 2\pi)$ so check if you look in a book.

What is $\text{Arg}(0)$? It could be anything from the definition, so we make the choice that $\text{Arg}(0) = 0$.

Definition: Let $\theta \in \mathbb{R}$ and define $e^{i\theta} = \exp(i\theta)$ to be the complex number $\cos(\theta) + i\sin(\theta)$.

So any complex number z with $|z| = 1$ can be written as $z = e^{i\theta}$ for some $\theta \in \mathbb{R}$. Any complex number z can be written as $z = re^{i\theta}$ for some $r \geq 0$ and $\theta \in \mathbb{R}$. This is commonly called the *polar form* of a complex number.

Let $z = a + bi = re^{i\theta}$. We can work out r as $r = |z| = \sqrt{\overline{z}z} = \sqrt{a^2 + b^2}$. Working out θ needs a bit of trigonometry: the naive formula is $\theta = \tan^{-1}(b/a)$. (Notice that $\tan^{-1}$ is the inverse function to tan. Sometimes this is written as arctan.) However, this will often give the wrong answer: for example, $\tan^{-1}(b/a) = \tan^{-1}((-b)/(-a))$, but $-a - bi = -z$ is different from z (unless $z = 0$). It often helps to sketch z on the Argand Diagram.



A sketch of $z = re^{i\theta}$.
Let $w = 2i - 1 = se^{i\varphi}$ for some $s \geq 0$ and $\varphi \in (-\pi, \pi]$ be the found. Then $s^2 = 2^2 + 1 = 5$ so $s = \sqrt{5}$. A bit of trigonometry shows that $\tan(\pi - \varphi) = 2/1 = 2$ so $\pi - \varphi = \tan^{-1}(2)$. Thus $\varphi = \pi - \tan^{-1}(2)$ (which doesn't simplify). Using "the formula" would give the wrong answer here!

[3]Look up in a dictionary the difference between "principal" and "principle"; I will probably confuse them at some point in lectures!

Let $z = re^{i\theta}$. On the Argand Diagram, we have that r is the distance from z to the origin 0, and that θ is the angle between the positive real axis and the line from 0 to z. So we can think about complex numbers in a more "geometric" way.

Claim: Let $\theta, \varphi \in \mathbb{R}$. Then $e^{i\theta}e^{i\varphi} = e^{i(\theta+\varphi)}$.

We prove this by using some trigonometry. The following is now easy to prove.

Claim: Let $z, w \in \mathbb{C}$ have polar forms $z = re^{i\theta}$ and $w = se^{i\varphi}$. Then $zw = rse^{i(\theta+\varphi)}$.

So if we multiplying two complex numbers, we multiply their lengths, but add their arguments. For example, in the Argand Diagram, multiplying by $e^{i\theta}$ is the same as rotating about the origin, by an angle of θ.

In the last workbook, we found the solutions to $|z - 1| = |z + 1|$. We can now think about this geometrically. For $z, w \in \mathbb{C}$, $|z - w|$ is the distance from $z - w$ to 0, or equivalently, the distance from z to w. So $|z - 1| = |z + 1|$ is exactly the statement that z is equidistant from 1 and -1. It's not too hard to see that this is simply all the z which are on the imaginary axis.

Definition: Let $z = a + bi \in \mathbb{C}$. We define $e^z = \exp(z)$ to be the complex number $e^a e^{bi} = e^a(\cos(b) + i\sin(b))$.

So we have extended the exponential function to the complex numbers. Notice that if $a \in \mathbb{R}$, then $a = a + 0i \in \mathbb{C}$ and $e^{a+0i} = e^a(\cos(0) + i\sin(0)) = e^a + 0i = e^a$, so our new definition is consistent with the old one.

Claim: For any $z, w \in \mathbb{C}$, we have that $\exp(z + w) = \exp(z)\exp(w)$.

Notice that $e^0 = 1$, so it follows that $e^z e^{-z} = 1$ for any $z \in \mathbb{C}$. That is, we have that $1/e^z = e^{-z}$. In particular, we have the useful formula

$$\frac{1}{r\exp(i\theta)} = \frac{1}{r}\exp(-i\theta).$$

Proof by Mathematical Induction

Quite often in mathematics, we meet problems which break down into one problem for each natural number, and in such a way that knowing the solution for $n \in \mathbb{N}$ allows you to work out the solution for $n + 1 \in \mathbb{N}$. We can solve such problems using *Mathematical Induction*.[4]

For example, by trying out some small numbers, we can convince ourselves that the following is probably true:

Claim: For each $n \in \mathbb{N}$, we have that $2^n/2 \leq n!$.

Remember that $n!$ is read as "n factorial" and means $n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$.

Indeed, we can draw up a table:

n	1	2	3	4	5
$2^n/2$	1	2	4	8	16
$n!$	1	2	6	24	120

To use Proof by Induction, we have to have a collection of claims, one for each $n \in \mathbb{N}$. We might call this $P(n)$. In our case, $P(n)$ is the claim that $2^n/2 \leq n!$. Notice that we **don't have that** $P(n) = 2^n/2 \leq n!$, which is a nonsense statement. We could write $P(n) =$"$2^n/2 \leq n!$", but I prefer to write a few words.

Firstly, we have to check that $P(1)$ is true. This is easy for us: if $n = 1$ then $2^n/2 = 1$ and $n! = 1$, so certainly $2^n/2 \leq n!$.

The second step is the clever bit. We *assume* that $P(n)$ holds, and using this information (and our general ingenuity) we then *prove* that $P(n + 1)$ holds. Notice that this will probably

[4] I shall often just say "induction", but this should not be confused with the philosophical meaning of induction, which is weaker (see Wikipedia).

be much easier than proving directly that $P(n+1)$ holds, because we are allowed to use the extra information that $P(n)$ is true.

If we can do this, then we know that $P(n)$ holds for all $n \in \mathbb{N}$. Why? Well, $P(1)$ holds. So setting $n = 1$, by using the second step, we know that also $P(2)$ holds. But then we can use the second step again (with $n = 2$) to see that $P(3)$ must hold. We can continue this indefinitely, showing that $P(n)$ must hold for whatever n we like. A colleague likens this to toppling dominoes: $P(1)$ is the first domino we knock over, and the second step shows that each domino will knock over the following domino.

Back to our example. So we assume that $P(n)$ holds (without knowing anything about n other than that $n \in \mathbb{N}$). This means that $2^n/2 \leq n!$. We now want to prove that $P(n+1)$ holds. Let's start with the left-hand side, $2^{n+1}/2 = \frac{2^n}{2} \times 2$. Using that $P(n)$ holds, we know that this is $\leq n! \times 2$. As $n \in \mathbb{N}$, we have that $n \geq 1$ so $n+1 \geq 2$. Hence $n! \times 2 \leq n! \times (n+1) = (n+1)!$. We might write this neatly as

$$\begin{aligned} 2^{n+1}/2 = 2 \times \frac{2^n}{2} &\leq 2 \times n! \qquad \text{because } P(n) \text{ holds, so } 2^n/2 \leq n!, \\ &\leq (n+1) \times n! \qquad \text{because } n+1 \geq 2, \\ &= (n+1)! \end{aligned}$$

So we have shown that $P(n+1)$ is true. By (mathematical) induction, we have shown that $2^n/2 \leq n!$ for any $n \in \mathbb{N}$.

The following is another way to write this, perhaps using notation similar to that which you used at school.

Claim: For every $n \in \mathbb{N}$, we have that $2^n/2 \leq n!$.

Proof: We use a proof by induction. We first prove the claim when $n = 1$. Then $2^n/2 = 1$ and $n! = 1$, so certainly $2^n/2 \leq n!$.

Assume the claim holds when $n = k$. We prove that it will hold when $n = k+1$. Indeed,

$$\begin{aligned} 2^{k+1}/2 = 2 \times \frac{2^k}{2} &\leq 2k! \quad \text{as the claim holds for } n = k, \\ &\leq (k+1) \times k! \quad \text{as } 2 \leq k+1, \\ &= (k+1)!, \end{aligned}$$

so the claim holds for $n = k+1$. By induction, the claim holds for all $n \in \mathbb{N}$. □

The following notation is useful, and will be used in lots of other mathematics modules.

Definition: Let $A \subseteq \mathbb{Z}$ and let $f : A \to \mathbb{C}$ (or $\mathbb{R}$ or $\mathbb{Z}$ etc.) be a function. Suppose that for $a, b \in \mathbb{Z}$ with $a \leq b$, we have that $a, a+1, a+2, \cdots, b-1, b \in A$. Then we write

$$\sum_{k=a}^{b} f(k) \quad \text{as shorthand for} \quad f(a) + f(a+1) + \cdots + f(b-1) + f(b).$$

This is probably most easily understood from some examples:

$$\sum_{k=1}^{10} k^2 = 1 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2 = 385,$$

$$\sum_{t=5}^{5} f(t) = f(5), \qquad \sum_{r=1}^{n} r = 1 + 2 + 3 + \cdots + n \qquad \text{for } n \in \mathbb{N},$$

$$\sum_{j=-1}^{2} |j| = |-1| + |0| + |1| + |2| = 1 + 0 + 1 + 2 = 4,$$

Here's another example of induction, written out without any words of explanation.

Claim: For $n \in \mathbb{N}$, we have that $\sum_{r=1}^{n} r = n(n+1)/2$.

Proof: For $n = 1$, we have that $\sum_{r=1}^{n} r = 1$, while $1(1+1)/2 = 2/2 = 1$, so the claim holds. Suppose that the claim holds for n. Then

$$\sum_{r=1}^{n+1} r = \Big(\sum_{r=1}^{n} r\Big) + (n+1) = \frac{n(n+1)}{2} + (n+1) \text{ using the induction hypothesis,}$$
$$= \frac{n^2+n+2n+2}{2} = \frac{(n+1)(n+2)}{2},$$

as required to show that the claim holds for $n+1$. So the claim is true by induction. □

Notice that I didn't use the notation $P(n)$, but instead referred to the "claim". The phrase "induction hypothesis" means that I was using that the assumption that the claim held for n. If you prefer to write $P(n)$, or use the form "Assume true for $n = k$, then prove true for $n = k+1$" then feel free to do so.

Ideas for the tutorial

Here are some ideas for things which might be interesting to discuss; "answers" will be put on the VLE.

➥Suppose we want to define $f : X \to \mathbb{R}$ by $f(x) = \sqrt{x^2+5x+6}$. What is the "maximal" set $X \subseteq \mathbb{R}$ on which this makes sense? Is the resulting map injective and/or surjective?

Let Y be the image of f. Let $g : X \to Y$ be the map f, considered now as having codomain Y. Find a subset $Z \subseteq Y$ such that the *restriction* of g to Z gives a bijection $Z \to Y$. (By "restriction", we mean the map $h : Z \to Y$ given by $h(z) = g(z)$ for each $z \in Z$). What is a "formula" for $h^{-1} : Y \to Z$?

➥Notice that each of the functions f, g, h above are defined using the "same formula", but as each has different domains or codomains, they are all different functions. They also behaved differently with respect to being injective and/or surjective: that's why we need to be so pedantic about domains and codomains.

➥Consider the functions

$$a : [-2, 8] \to [2, 32];\ a(x) = 3(x+2)+2, \qquad f : [0,\infty) \to [3,\infty);\ f(x) = x^2+2x+3,$$

In each case, *prove* that the functions are bijections, and write down the inverse functions.

➥There are loads of similar examples of varying difficulty (throw in a few sin or cos functions for fun). Seeing lots of "quick proofs" of injectivity and surjectivity is good.

➥Investigate the following claim: For any $x \in \mathbb{Z}$ and $n \in \mathbb{N}$, when $x \neq 1$, we have that $(x^n-1)/(x-1)$ is an integer. What's a general formula for $(x^n-1)/(x-1)$? Prove this by induction on n.

➥Let $x_1, \cdots, x_n \in [0,\infty)$. Prove (by induction) that

$$\frac{x_1+x_2+\cdots+x_n}{n} \geq \big(x_1x_2\cdots x_n\big)^{1/n}.$$

➥I have put some "optional" questions at the end: these provide further practise of complex number manipulation and induction. These could be attempted by students now, or before revision, or maybe even covered in tutorials. The final induction is pretty tricky, and might be good to discuss in a group.

Problem Set 3

Due in at the lecture on Wednesday 3 November.

Remember that $\cos(\pi/4) = 1/\sqrt{2} = 2^{-1/2}$ and so forth. For other angles, leave answers in the form $\sin^{-1}(1)$, not $1.570\cdots$, and so forth.

1. (a) Write the following complex numbers in the form $a + bi$ for some $a, b \in \mathbb{R}$.

   i. $\cos(\pi/6) + i\sin(\pi/6)$
   ii. $\sqrt{2}e^{i\pi/4}$
   iii. $4e^{i\pi}$

   (b) Find the modulus and **principal** argument of the following complex numbers. For each one, give a quick sketch of where the number is on the Argand Diagram.

   i. i
   ii. $1 - i$
   iii. $11 + 60i$
   iv. $\sin(2) + i\cos(2)$ (this is tricky!)

2. (a) Decide if each of the following functions are injective, or not, and/or surjective, or not. In each case, make a quick sketch of the function, and use this to make your decision: write a *few* words of explanation.

   i. $f : \mathbb{R} \to \mathbb{R}; f(x) = x + |x|$

   ii. $g : [0, \infty) \to [0, \infty); g(a) = \begin{cases} a^2 & \text{for } 0 \le a < 1, \\ a^2 - 1 & \text{for } 1 \le a. \end{cases}$

   iii. $h : [0, \infty) \to \mathbb{R}; g(a) = \begin{cases} a^2 & \text{for } 1 \le a, \\ a^2 - 1 & \text{for } 0 \le a < 1. \end{cases}$

   (b) Decide if each of the following functions are injective, or not, and/or surjective, or not. In each case, either give a quick proof that the property holds, or give a counter-example. (You can still draw a sketch if you wish!)

   i. $f : \{a, b\} \to \{1, 10\}$ given by $f(a) = 10$ and $f(b) = 1$.

   ii. $h : [0, \infty) \to [0, \infty); h(x) = \begin{cases} x & \text{for } 0 \le x < 5, \\ x + 1 & \text{for } 5 \le x. \end{cases}$

   iii. $j : \mathbb{C} \to \mathbb{C}$ given by $j(z) = e^i z$.

   (c) The following functions are bijective. Describe, in as simple a way as possible, the inverse function.

   i. $\alpha : \{a, b\} \to \{1, 10\}$ given by $\alpha(a) = 10$ and $\alpha(b) = 1$.
   ii. $\beta : \mathbb{Z} \to \mathbb{Z}; \beta(m) = -m$;
   iii. $\gamma : \mathbb{R}^2 \to \mathbb{R}^2$ given by $\gamma(a, b) = (a, a + b)$.

3. Let $f : X \to Y$ be a function such that there exists $g : Y \to X$ with $f \circ g = \mathrm{id}$ and $g \circ f = \mathrm{id}$. Prove that f is a bijection.

   Hint: You have to show both that f is injective, and that f is surjective. You will have to use the function g to do this. Look on page 2 and see how we showed that $f : \mathbb{R} \to (0, \infty); f(x) = e^x$ was a bijection.

4. Prove the following by using mathematical induction.

   (a) Prove that $\sum_{j=1}^{n}(2j - 1) = n^2$ for any $n \in \mathbb{N}$.
   (b) Prove that $2n \le 2^n$ for any $n \in \mathbb{N}$.

(c) Prove that $\sum_{r=1}^n r^3 = \left(\sum_{r=1}^n r\right)^2$ for any $n \in \mathbb{N}$. *Hint: In the notes, we gave a formula for* $\sum_{r=1}^n r$. *It might help to write down what you hope to get, and "work backwards".*

(d) Let $r \in \mathbb{C}$ with $r \neq 1$. Show that $\sum_{j=1}^n r^j = (r - r^{n+1})/(1-r)$ for any $n \in \mathbb{N}$.

Hand in your work with **your name** and **your tutor's name** (one of Dr Daws, Prof Martin, Prof Read, Prof Bielawski, Mr Mihiesi (Naz) or Ms Montalvo-Ballesteros (Mayra)) and **your tutorial time** (either 9am or Noon) written clearly at the top.

Here are some further, optional questions, which give some extra practise with complex numbers and induction. I'll give out model answers, so you can check your own work.

1. Write the following complex numbers in the form $a + bi$ for some $a, b \in \mathbb{R}$.

   (a) $4(\cos(\pi) + i\sin(\pi))$

   (b) $(5e^{2i})^{-1}$

   (c) $2^{i\pi/3} + e^{i\pi/4}$

2. Find the modulus and **principal** argument of the following complex numbers.

   (a) $-i$

   (b) $1 + i$

   (c) $-4 + 3i$

   (d) $\cos(\theta) - i\sin(\theta)$ when $\theta \in [0, \pi]$.

3. Write the following complex numbers in the form $r\exp(i\theta)$ for some $r \in [0, \infty)$ and $\theta \in \mathbb{R}$.

   (a) 2

   (b) $-i$

   (c) $11 + 60i$

   (d) $3 - 4i$

   (e) $\exp(i)$

   (f) $\exp(2 + 3i)$

   (g) $\exp(z)$ where $z \in \mathbb{C}$

   (h) $(se^{i\varphi})^2$ where $s \in [0, \infty)$ and $\varphi \in \mathbb{R}$

4. Prove the following by using mathematical induction:

   (a) Prove that $\sum_{r=1}^n r^3 = \left(\sum_{r=1}^n r\right)^2$ for all $n \in \mathbb{N}$. *Hint: In the notes, we gave a formula for* $\sum_{r=1}^n r$. *It might help to write down what you hope to get, and "work backwards".*

   (b) Prove that $6^n - 1$ is divisible by 5 for all $n \in \mathbb{N}$. *Hint: An equivalent, but possibly easier, way to write this: For each* $n \in \mathbb{N}$, *there exists some* $k \in \mathbb{N}$ *with* $6^n - 1 = 5k$.

   (c) Fix some $x \in [0, \pi/2]$. Prove that $\sin(nx) \leq n\sin(x)$ for all $n \in \mathbb{N}$.