MATH1035: Workbook Two

–M. Daws, 2010

Solving equations involving complex numbers

We can split a complex number up into two real numbers in a useful way:

Definition: For a complex number $z = a + bi$, the *real part* of z is a, and the *imaginary part* of z is b (notice: not bi). We write $\mathrm{Re}(z) = a$ and $\mathrm{Im}(z) = b$. (Some books will use $\Re(z)$ and $\Im(z)$, but I cannot write these!)

This allows us to re-write problems involving a complex number as a problem involving two real numbers, which we can then usually solve. For example, suppose we wish to find all the solutions to $z^2 = i$, where $z \in \mathbb{C}$. Let $z = a + bi$, so that $z^2 = (a^2 - b^2) + 2abi$. Then

$$z^2 = i \implies \mathrm{Re}(z^2) = 0, \mathrm{Im}(z^2) = 1 \implies a^2 - b^2 = 0, 2ab = 1.$$

So $a^2 = b^2$, and taking the square-root, we get $a = b$ or $a = -b$ (remember the negative square root!) If $a = b$, then $1 = 2ab = 2a^2$ so $a = 1/\sqrt{2}$ or $a = -1/\sqrt{2}$. If $a = -b$ then $1 = 2ab = -2b^2$ which has no solutions (as b is a real number!) So we find

$$z = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \quad \text{and} \quad z = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i.$$

Ah! But we only used "$\Rightarrow$", so we should check that these really are solutions. Well,

$$\left(\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}i\right)^2 = \tfrac{1}{2} - \tfrac{1}{2} + 2\tfrac{1}{2}i = i,$$

as required (as if $z^2 = i$, surely also $(-z)^2 = i$). Notice that, as for real numbers, we always get a positive and negative square-root.

Example: find the solutions to $|z + 1| = 2$. Remember from before that for a complex number w,

$$|w| = \sqrt{w\overline{w}} = \sqrt{c^2 + d^2} \text{ if } w = c + di.$$

So it might help to solve $|z + 1|^2 = 4$ instead. Again, we let $z = a + bi$, so that

$$|z + 1|^2 = (a + 1)^2 + b^2.$$

Thus $|z + 1|^2 = 4$ implies that $(a + 1)^2 = 4 - b^2$, that is

$$a = -1 \pm \sqrt{4 - b^2}.$$

Is that our answer?

No, because a is meant to be a real number, so we need $4 - b^2 \geq 0$ in order to get a real square-root. So we need $b^2 \leq 4$, which is equivalent to $-2 \leq b \leq 2$. (If you don't see that immediately, think about it!) For any b in this range, we have two possibilities for a:

$$a = -1 + \sqrt{4 - b^2} \text{ and } a = -1 - \sqrt{4 - b^2}.$$

Again, a quick check shows that these are all solutions.

We'll see later in the course that there are more elegant (but also more complicated) ways to solve problems like this. If you haven't seen complex numbers before, I think it's good practice to do some of these problems the long way.

Sets

A *set* is a collection of objects. We list the objects in a set by using curly brackets: { and }. For example,

$$A = \{\text{All mathematicians}\}, \quad B = \{1, 2, 3, \cdots\}, \quad C = \{a, b, c\}.$$

Here B is just the natural numbers $\mathbb{N}$, C is a collection of letters (maybe these are mathematical variables?) and A is a collection of people (notice that it's pretty unclear what A is: who, exactly, is a mathematician?)

A set can contain other sets! This occurs in real-life: a shopping trolley might contain a number of shopping bags. We can think of each bag as being a set, as it contains various items of shopping. A mathematical example is the following:

$$A = \{5\}, \quad B = \{\{5\}, 6\}, \quad C = \{5, 6\}.$$

The contents of these sets are:

- A contains the number 5.
- C contains the numbers 5 and 6.
- B contains the number 6, and also contains A (the set containing 5).

This means that B and C *are different!*

The objects, or *elements*, of a set are the things it contains. For example, if $A = \{1, 2, 5\}$ then the elements of A are $1, 2$ and 5. Notice the difference: $A = \{1, 2, 5\}$ but the elements are 1, 2 and 5. We *don't* say "The elements of A are $\{1, 2, 5\}$" (well, it would probably be understood what was meant, but try to avoid this!)

We write

$$x \in A \text{ to mean that } x \text{ is an element of the set } A\text{, or "}x \text{ is a member of } A\text{".}$$

As usual, we write $x \notin A$ to mean that x is not a member of A. So with $A = \{1, 2, 5\}$, we have that $1 \in A$ but $3 \notin A$.

We can write more complicated sets in the form

$$\{x \in A : \text{Some condition on } x\}.$$

This is the set of all objects x which are in A (the "$x \in A$" part) which satisfy the condition. For example,

$$\{n \in \mathbb{N} : n \text{ is even, and } n \geq 10\}$$

is the set $\{10, 12, 14, 16, 18 \cdots\}$. The reason for using the more complicated expression is that it's completely water-tight– there can be no confusion about what is meant– whereas if I write $\{10, 12, \cdots\}$, you have to guess what the "$\cdots$" means. As things get more complicated, this will only become more true.

Definition: We write $\emptyset$ for the set which contains nothing: called the *empty set*.

This might seem odd, but it comes up all the time:

$$A = \{\text{People who are naturally Green haired}\}, \quad B = \{x \in \mathbb{N} : x^2 = 3\}$$

both contain nothing, and hence we have that $A = \emptyset$ and $B = \emptyset$. Notice also that $A = B$.

(Something which students sometimes write is "The solutions are $\emptyset$". This is nonsense, unless you except the answer to be a set! What the student usually meant to say is "The set of solutions is $\emptyset$". For example, we could say "The set of solutions to $x^2+2x+1=0$ is $\{-1\}$" which is the same as "The solution to $x^2+2x+1=0$ is $x=-1$". Similarly, we could say "The set of real solutions to $x^2=1$ is $\emptyset$", but I much prefer "There are no real solutions to $x^2=1$").

The following is a very common piece of notation, so we introduce it here.
Definition: Let $x, y \in \mathbb{R}$ with $x \leq y$. Then we define

$$\begin{aligned}
[x, y] &= \{t \in \mathbb{R} : x \leq t \leq y\}, \\
(x, y) &= \{t \in \mathbb{R} : x < t < y\}, \\
[x, y) &= \{t \in \mathbb{R} : x \leq t < y\}, \\
(x, y] &= \{t \in \mathbb{R} : x < t \leq y\}.
\end{aligned}$$

So $[x, y]$ is all the real numbers between x and y, including x and y, while (x, y) is all the real numbers between x and y, not including x and y. We could also say that (x, y) is the collection of all real numbers *strictly* between x and y.

Operations on sets

Definition: Let A and B be sets. We write $A \cap B$ for the *intersection* of A and B. This is the collection of objects in both A and B. We write $A \cup B$ for the *union* of A and B. This is the collection of objects either in A, or in B, or in both.

(Aside: Mathematicians use "or" to me one thing, or the other, or maybe both. This differs from normal English: "I'll come to the pub, or I'll stay in and do my maths homework" would generally not mean that you'd do both things).

For example, if

$$A = \{1, 2, 3, 4, 5, 6\}, \qquad B = \{3, 5, 7\},$$

then

$$A \cap B = \{3, 5\}, \qquad A \cup B = \{1, 2, 3, 4, 5, 6, 7\}.$$

Definition: Let A and B be sets. We write $A \setminus B$ for the *set difference* of A and B. This is the collection of all objects which are in A but not in B. Notice the difference between "$\setminus$" and "$-$".

With our example above,

$$A \setminus B = \{1, 2, 4, 6\}.$$

We can draw *Venn Diagrams* to help to get a feel for these operations; you probably saw such things at school, or in the Probability course. In this course, I will be using simple properties about sets as an excuse to carefully explain various techniques of proofs. As such, I will *not* regard Venn Diagrams as being proofs. But drawing a Venn diagram can be useful to get some intuition, before you write the formal proof.

Relationships between sets

Definition: Let A and B be sets. We say that A and B are *equal* if they contain the same objects. We say that A is *contained in* B, written $A \subseteq B$, if every object in A is also in B. When this is not true, we write $A \not\subseteq B$.

For example, let

$$A = \{1, 2, 3, 10\}, \quad B = \{5, 6, 7\}, \quad C = \{1, 2, 3, 5, 6, 7, 8, 10\}.$$

Then $A \subseteq C$, $B \subseteq C$ but $A \nsubseteq B$, $B \nsubseteq A$ and so forth.

Definition: Let A and B be sets. We say that A is *strictly contained* in B if $A \subseteq B$ but $A \neq B$. We write $A \subsetneq B$ or $A \subsetneqq B$. A little thought will show that this means that every element of A is also an element of B, but, also, that there are elements of B which are not in A.

With the above example, we actually have that $A \subsetneq C$ and $A \subsetneq B$.

(Aside: Some authors use the symbol $\subset$. I find this confusing, as it's hard to remember if it means $\subseteq$ (allowing the possibility of equality) or if it means $\subsetneq$ (not allowing equality). All meanings exist, so if you are reading a book, be careful!)

More on proof

A common task in mathematics is to show that two objects which have different definitions are actually equal. For example, consider the sets

$$A = \{n \in \mathbb{N} : n \text{ is divisible by } 4\}, \qquad B = \{4k : k \in \mathbb{N}\}.$$

Are these the same, or not? They don't look the same as written.

Claim: A natural number n is divisible by 4 if and only if $n = 4k$ for some natural number k.

To prove this, we will deploy the following strategy. We see that there is an "if and only if" in the claim. So we break the proof down into two parts. Firstly we suppose that n is divisible by 4, and show that $n = 4k$ for some k. This is the "only if" part. Then we suppose that all we know is that $n = 4k$ for some k, and proceed to show that n is divisible by 4. This is the "if" part.

(At this point, in this example, it's really easy to see that this is true. But in more complicated examples, this is just the first step!)

Proof: Suppose that $n \in \mathbb{N}$ is divisible by 4. This means that $n/4$ is a whole number, say $k = n/4$. Thus $n = 4k$, and $k \in \mathbb{N}$.

Conversely, if $n = 4k$ for some $k \in \mathbb{N}$, then $n/4 = k$, which is a whole number, so n is divisible by 4. □

Notice the use of the word "conversely". This signals to the reader that we have finished proving one implication (in this case, the "only if" or $\Rightarrow$ implication) and that we are about to begin proving the other implication.

The following might seem obvious (perhaps after a bit of thought) but is very useful.

Claim: Let A and B be sets. Then $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

Notice how this is also a "divide and conquer" strategy. Instead of showing that $A = B$ directly, we split the problem into two, easier parts: first show that $A \subseteq B$, and then show that $B \subseteq A$. So the proof will look something like:

- First show $A \subseteq B$. Do this by:
  - Let $x \in A$ by arbitrary (so we don't know anything about x, except that it's in A).
  - Come up with some argument to show that $x \in B$ (obviously this part depends on the specific problem).
  - So we've shown that every element of A is an element of B.
- Now (possibly using a completely new argument) show that $B \subseteq A$.

Here is an example.

Claim: For any sets A, B and C,

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Proof: We first show that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$. Let $x \in A \cap (B \cup C)$. So we know that $x \in A$ and $x \in B \cup C$. That is, we know that $x \in A$ and $x \in B$, or $x \in A$ and $x \in C$. That is, $x \in (A \cap B) \cup (A \cap C)$.

We now show that $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. Let $x \in (A \cap B) \cup (A \cap C)$. So $x \in A$ and $x \in B$, or $x \in A$ and $x \in C$. So in either case $x \in A$, and also, either $x \in B$ or $x \in C$. Thus $x \in A \cap (B \cup C)$.

We conclude that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. □

Writing the final line in the proof is optional, but it's nice to signal to the reader that you've finished the proof.

Functions

A *function* from a set X to a set Y is an assignment of each element in X to an element in Y. We write $f(x)$ for the element in Y which f associates to $x \in X$. We can also call f a *mapping*, and we say that f maps from X to Y. This is written as $f : X \to Y$.

You might previously have considered, for example,

$$f(x) = x^2 + 5x + 7$$

to be a function. We wish to be more precise, and to specify exactly what the set f maps from and what set f maps to. So we might say

$$f : \mathbb{R} \to \mathbb{R} \quad \text{given by} \quad f(x) = x^2 + 5x + 7.$$

This tells us that we should regard f as mapping from $\mathbb{R}$ to $\mathbb{R}$. Notice that $x^2 + 5x + 7 \geq 0$ for all $x \in \mathbb{R}$, so we also have

$$g : \mathbb{R} \to [0, \infty) \quad \text{given by} \quad g(x) = x^2 + 5x + 7.$$

Formally, g and f *are not* the same function. Similarly,

$$h : \mathbb{Z} \to \mathbb{N} \quad \text{given by} \quad h(x) = x^2 + 5x + 7,$$

is yet another, different, function.

It is also important to realise that not all functions are given by a simple formula. For example, we could define $\alpha : \mathbb{N} \to \{0, 1\}$ where $\alpha(n) = 1$ if n is a square number, and $\alpha(n) = 0$ otherwise. So

$$\alpha(1) = 0, \quad \alpha(4) = 1, \quad \alpha(9) = 1, \quad \alpha(11) = 0,$$

and so forth. But there is no formula which gives α.

However, α certainly has an *algorithm* which defines it. For any value of n, at least in principle, we can find what $\alpha(n)$ is. Not all functions are even like this; but we shall not pursue this further here.

Definition: Let $f : X \to Y$ be a function. The *domain* of f is X; this is the set which f maps from. The *codomain* of f is Y, the set which f maps to. I prefer the term *target*, but this is not often used.

For $x \in X$, we call $f(x)$ the *value of f at x* or the *image of x under f*.

It is very useful to be able to define a function by piecing together a number of functions which are defined by formula. For example, we define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 & \text{for } x \leq 0, \\ x & \text{for } x > 0. \end{cases}$$

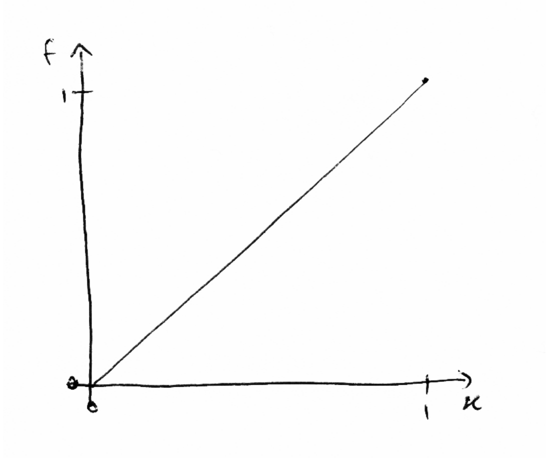
We interpret this as follows: first, pick $x \in \mathbb{R}$, and consider how to find $f(x)$. If $x \leq 0$, then we have that $f(x) = x^2$. If $x > 0$, then we have that $f(x) = x$. As any real number is either ≤ 0 or > 0, f is defined on all of $\mathbb{R}$. Notice that we have not defined two functions here: we have one function, which is defined by two different formulae. Similarly, above we could have written α as

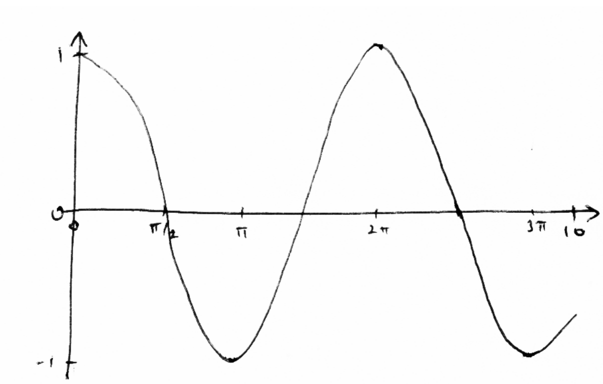
$$\alpha : \mathbb{N} \to \{0,1\} \text{ given by } \alpha(n) = \begin{cases} 1 & \text{when } n \text{ is a square number,} \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $f : X \to Y$ is a function, where $X, Y \subseteq \mathbb{R}$. Being able to sketch such a function is a useful skill. For example, consider

$$f : [0,1] \to \mathbb{R} \quad \text{given by} \quad f(x) = x,$$
$$g : [0,10] \to \mathbb{R} \quad \text{given by} \quad g(x) = \cos(x).$$

Here are two attempts at sketching these functions: notice that they are not totally accurate, but, hopefully, indicate the general structure of these maps.





Greek letters

Mathematicians often use Greek letters, when we run out of Latin ones. You got a handout in Intro Week listing them all, but here are some common ones: α is Alpha, β is Beta, γ is Gamma, δ is Delta and ϵ (or ε) is Epsilon.

Ideas for the tutorial

As usual, a reminder that what happens in the tutorial should be down to the tutor and the tutees. Here are some ideas for things I am *guessing* might be interesting to discuss. It's possible that I will have mentioned some of these in lectures. I'll upload "answers" to the VLE, in case your tutor group doesn't discuss everything (which is very likely, as there is a lot here).

➥Think about the difference between

$$A = \{\{5\}, 6\} \quad , \quad B = \{5, 6\} \quad \text{and} \quad C = \{5\}.$$

What are the elements of A? Is $C \subseteq A$?

➥Express each of the following sets as a list:

- $\{x \in \mathbb{Z} : -2 \leq x \leq 2\}$
- $\{x \in \mathbb{C} : x^2 + x + 1 = 0\}$
- $\{n \in \mathbb{Z} : n^2 = 9\}$
- $\{x \in \mathbb{R} : x^2 + x + 1 = 0\}$

➥Again, the fact that sets can contain sets can cause confusion. Is it true, or not, that

$$\{\emptyset\} = \emptyset?$$

➥Could discuss Venn Diagrams, although I'm keen to avoid these except as a useful visualisation tool.

➥Stress the fact that functions are more than just formulae, and come with domains and codomains defined. Does the following make sense?

$$f : \mathbb{C} \to \mathbb{R}; \quad f(z) = \begin{cases} \operatorname{Im}(z) & \text{when } \operatorname{Re}(z) \geq 0, \\ -\operatorname{Im}(z) & \text{when } \operatorname{Re}(z) \leq 0. \end{cases}$$

The following does make sense:

$$g : \mathbb{N} \to \mathbb{N}; \quad \text{given by} \quad g(n) = \text{distint prime factors of} n.$$

So $g(1) = 1$ and $g(2) = 1$. What's the smallest n with $g(n) = 2$? What about $g(n) = 3$? What is $g(1000)$?

➥This is rather a *long* question, but is fun, and in the spirit of the optional Question 6 below. Find

$$\{4m + 1 : m \in \mathbb{N}\} \cap \{5n + 2 : n \in \mathbb{N}\}.$$

We *don't* have tools like modular arithmetic. Experiment a bit! Form a conjecture! Prove this conjecture!

➥After the homework deadline, it might be good to think about Question 6 (see below) in the tutorial– I don't necessarily expect tutors to mark any attempts.

Problem Set 2

Due in at the **lecture** on Wednesday 20 October.

When you first get this sheet, we won't have covered everything in lectures yet. However, you should know enough to get start with Question 1, for example. It's in your best interest to work on the sheet over all the time before the deadline, and not to try to do everything at the last moment.

1. (a) Find all possible values of $a, b \in \mathbb{R}$ which satisfy the following equations.

   i. $a + bi = 2i - 5$

   ii. $a + bi = ai + b$

   iii. $(a + bi)^2 = (a - bi)^2$

   iv. $\dfrac{a + bi}{a - bi} = i$

   (b) By splitting into real and imaginary parts, or otherwise, find the solutions to the following equations, supposing that $z \in \mathbb{C}$. (You can use any method you like to solve these problems, as long as you *clearly state* what you are doing.)

   i. $|z - 1| = |z + 1|$

   ii. $z^2 = (\overline{z})^2$

   iii. $z^3 = -1$ (Is there an obvious solution? Does this help you? Think: Factorisation!)

2. (a) Write each of the following sets as a list $\{a, b, c, \cdots\}$.

   i. $A = \{n \in \mathbb{Z} : -5 \leq n \leq 3\}$

   ii. $B = \{n \in \mathbb{N} : -1 \leq n \leq 1\}$

   iii. $C = \{x \in \mathbb{R} : x^2 + 5x + 6 = 0\}$

   iv. $D = \{x \in \mathbb{N} : x^2 + 5x + 6 = 0\}$

   v. $E = \{x \in \mathbb{N} : x^3 - x = 0\}$

   (b) What are the objects (or elements) contained in the following sets? (This is very similar to the previous question, but requires you to use slightly different notation: read page 2 above.)

   i. $\alpha = \{1, 2, 3, \{4, 5\}, \{6, \{7, 8\}\}\}$

   ii. $\beta = \{x \in \mathbb{N} : x^2 = 5 \text{ or } x^2 = 4\}$

   iii. $\gamma = \{n + (1/n - 6) : n \in \mathbb{Z}, n \neq 0, \text{ and } -2 \leq n \leq 3\}$

   (c) Using the definitions of part (a), decide if the following are true or not (you need not give reasons):

   i. $A = E$

   ii. $B \subseteq E$

   iii. $D \subseteq C$

   iv. $C \subsetneq A$

   v. $B \subsetneq E$

   (d) Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{2, 4, 6, 8, 10\}$. Write each of the following as a list $\{a, b, c, \cdots\}$.

   i. $A \cap B$

   ii. $A \cup B$

   iii. $A \setminus B$

   iv. $(A \setminus B) \cup (B \setminus A)$

3. Let A, B and C be sets. Prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

   Hint: We proved something similar in lectures: how did we do so? Read the section above on "divide and conquer" strategies. You could draw a Venn diagram, or pick some simple examples for A, B and C to get a feel from the problem: but these would not be a proof.

4. Prove that for any $x, y \in \mathbb{Q}$, with $x < y$, we can find some $z \in \mathbb{Q}$ with $x < z < y$.

   Hint: What is this asking? Try some numbers: pick some rational numbers x and y with $x < y$ and try to find a suitable z. What if $x = 1$ and $y = 2$? What if $x = 1/2$ and $y = 3/4$? Is there some *systematic* way to choose z? Try to write *something* down as evidence to your tutor that you have thought about this.

5. Sketch the following functions: you need *not* prove why your sketch is of the correct form.

   (a) $f : [0, 1] \to \mathbb{R}$ defined by $f(x) = x^2$

   (b) $f : [-1, 1] \to \mathbb{R}$ defined by $f(x) = x$, for $-1 \le x \le 0$, and $f(x) = x^2$ for $0 \le x \le 1$

   (c) $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{for } 0 \le x < 1/3, \\ 1 & \text{for } 1/3 \le x < 2/3, \\ 3(x - 1/3) & \text{for } 2/3 \le x \le 1. \end{cases}$$

6. [Optional, does not count, but might be interesting] This question will lead you towards a proof of the following claim: "The number 300 000 067 110 605 737" is not a perfect square."

   (a) Write down what it means for a number to be a "perfect square". (Notice that the number here is much too large to allow us to use a calculator! There do exist computer programs which can handle such big numbers, but you should definitely worry about how the computer actually does this. As this is not a computer science course, we'll come up with an argument which doesn't need a computer.)

   (b) Let $X = 300000067110605737$. So the question is asking us: show that for any choice of $n \in \mathbb{N}$, we *do not* have that $n^2 = X$. For some small values of n, write down what n^2 is.

   (c) Look at the last digit of n^2: do you notice any patterns? Write down what you notice.

   (d) If n is even, what is the last digit of n^2? Write a proof that if n is even, we cannot have that $n^2 = X$.

   (e) Now concentrate upon when n is odd. Do you notice any pattern about the final digit of n^2 when n is odd? Prove that this pattern is always true.

   (f) You should now have enough to prove the original claim.

   (g) Finally, try to write down a *short* and *formal* proof of the original claim.

Hand in your work with **your name** and **your tutorial group or tutor's name** clearly marked.