MATH1035: Workbook One

–M. Daws, 2010

This course is MATH1035, "Analysis". The course is split in two, with two different lecturers. The 2nd half of the course will be given by Prof Dales, and concentrates upon what a mathematician calls "analysis", which to quote Wikipedia, "has its beginnings in the rigorous formulation of calculus." Prof Dales will start lecturing in week 9 (or maybe 10).

The first half of the course is given by Dr Daws (the author of these notes). My half will really be an introduction to "pure mathematics". That is, I will cover some fundamental ideas which underpin the more abstract areas of mathematics: number systems, sets, functions, relations, methods of proof and so forth. Some of these will be rather like school mathematics (for example, performing calculations with complex numbers) but some will be very different: thinking about what a "function" is, which will probably take some time to sink in! Above all, don't be discouraged if it doesn't all make sense immediately.

Some practical matters:

- You will have two lectures a week, (provisionally) Monday, 2-3pm, LT22 and Wednesday, 1-2pm, LT22. I will not produce printed lecture notes: you will get these "workbooks", but it's an important skill to learn how to take notes in a maths lecture. It is important also to learn to *think about the material* as you go along!
- You will have a weekly "tutorial" in a small group (about 12-14 students) on Friday morning. Everyone has been automatically allocated to a group: see your personalised timetable on the Portal. The tutorial is an opportunity to ask questions, and to work through examples, in small groups, with your tutor. At the back of this workbook I have listed some possible things to do in the tutorials. But, you should negotiate with your tutor exactly what you will do: your tutor is there to help!
- There will be fortnightly example sheets set: these should be handed in at the Wednesday lecture. They will count (a small amount) towards your final grade, and will be marked by your tutor. Your tutor should return your marked answers, with feedback, in a timely manner.

This "workbook" contains:

- Skeleton notes; but not *everything* that is important;
- Mini-exercises to be thought about on your own, or in the "tutorial";
- Exercises to be completed (on another piece of paper!) and handed in: these count (a small amount) to your final grade.

The workbooks allow me to re-iterate important points from lectures, and to give you an indication of the important key points. The mini-exercises allow you time to think about concepts from lectures. I also hope that you will discuss some of the mini-exercises with your tutor.

The main exercises are for practise, and also to, yet again, get you to *think* about the material in lectures. I cannot stress this enough: there is very little in this course to just learn: most of the course is about how to think in a slightly different way from what you are used to.

Definitions

A *definition* in mathematics is giving a very precise meaning to a word or phrase. Comparing this with the English meaning of "definition" can be tricky: a mathematical definition means nothing more, and nothing less, than what it says.

In lectures, we wrote i for the "number" which when squared gives -1. I *did not* make this a formal definition, as it's unclear if, really, such a number exists! In this course, we shall tactically ignore this problem.

Indeed, you might (and probably *should*) worry that perhaps this will get us into trouble: maybe if we were clever enough, we could deduce nonsense. We call this a *contradiction*, and it would be bad news! Rest assured that this doesn't happen, in this case. We saw in lectures why it is impossible to define $1/0$: this always does lead to a contradiction.

Number systems

I want to introduce some standard types of numbers:

$$\begin{aligned}
\mathbb{N} &= \text{The natural numbers} = \{1, 2, 3, 4, \cdots\},\\
\mathbb{Z} &= \text{The integers} = \{\cdots, -2, -1, 0, 1, 2, 3, \cdots\},\\
\mathbb{Q} &= \text{The rationals} = \{a/b : a, b \in \mathbb{Z}, b \neq 0\},\\
\mathbb{R} &= \text{The real numbers} = \{???\}.
\end{aligned}$$

Some words of explanation are needed here! Firstly, we write $\{\cdots\}$ to indicate a collection of objects (technically, this is a "set", but more on this later). By $\{1, 2, 3, 4, \cdots\}$ I mean by the "$\cdots$" that we continue forever: so this is the same as $\{1, 2, 3, 4, 5, 6, 7, 8, \cdots\}$. To define $\mathbb{Q}$, I put a formula inside the curly brackets, then a colon, and then listed some constraints on the numbers. So in words

$$\mathbb{Q} = \{\text{All numbers of the form } a/b \text{ where } a \text{ and } b \text{ are integers, and } b \text{ is not zero }\}.$$

Again, more on this notation later. I have not defined the real numbers– they are just the sort of numbers you use to "measure things"; they are "decimal numbers". In Prof. Dales's part of the course, you will see a proper, mathematical definition.

A final word of warning: some authors (even some colleagues of mine) think that $\mathbb{N} = \{0, 1, 2, 3, \cdots\}$. Unfortunately, this happens in mathematics: we cannot quite agree on definitions. But[1] all that matters is that we are *clear* in our definitions, and that we are *consistent*.

Complex numbers

A *complex number* is a number of the form $a+bi$ where a and b are real numbers. Remember that i is a number which we have basically invented, and all we know is that $i^2 = -1$. You can manipulate complex numbers by treating i as a "variable" or "indeterminate", remembering that $i^2 = -1$. So, for any real numbers a, b, c and d,

$$(a+bi) + (c+di) = (a+c) + (b+d)i, \quad (a+bi) - (c+di) = (a-c) + (b-d)i.$$

Multiplying is similar,

$$(a+bi)(c+di) = ac + bic + adi + bidi = (ac-bd) + (bc+ad)i.$$

Division requires a little "trick",

$$\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{(ac+bd)+(bc-ad)i}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i.$$

[1] And maybe this is a slightly controversial opinion, but it's what I believe. It doesn't matter if 0 is considered a natural number of not– it's just convenient in this course that 0 is not.

Did you follow that just by reading? If not, *check my workings!*[2]

Definition: Let $z = a + bi$ be a complex number. The *complex conjugate* of z, written as $\overline{z}$ (or maybe z^*), is defined as $a - bi$.

We proved in lectures that $z\overline{z}$ is always a positive real number, so we can take the square-root, defined to be $|z|$, called the *modulus* of z. So

$$|z| = \sqrt{z\overline{z}}, \qquad |z|^2 = z\overline{z}$$

Thinking about proof

A "proof" is a rigorous mathematical argument. We start with *hypotheses* and proceed to deduce *conclusions*.

Claim: There are no integer solutions to $x^2 = 5$.

If a claim is written in the form "**If** something, **then** something" then it is easy to find the hypotheses and the conclusions. But the claim above is of a different form, so we first re-write it:

Claim: If $x^2 = 5$, then x is not an integer.

It might[3] take a moment to see that these really are two ways of saying the same thing. Often (but not always) it helps to write a claim in the "If something, then something" format, as then you see where to start. So, to find a proof, we need to start by supposing that $x^2 = 5$, and then come up with an argument that x cannot be an integer.

Here is a "proof" of this claim:

$$\boxed{x^2 = 5 \;\Rightarrow\; x = \pm\sqrt{5} \text{ not an integer.}}$$

This is *not* a good proof for many reasons: I have not said what I am doing; I have not written in sentences; I have used symbols where maybe words would have been clearer. Look in your booklet "How to write mathematics" for further hints.

The symbol $\Rightarrow$ means "implies". Let A and B be statements. Then we would say $A \Rightarrow B$ as "A implies B" or "B if A" or "if A, then B" or "whenever A, then B" or "A only if B" (this last one requires a bit of thought! Try A="I have a son" and B="I have a child" for example.) We write $A \Leftarrow B$ as an alternative for $B \Rightarrow A$. We write $A \Leftrightarrow B$ as shorthand for "$A \Rightarrow B$ *and* $B \Rightarrow A$"; this is often written as "A *if and only if* B". Remember this shorthand: if you need to prove that $A \Leftrightarrow B$, then often the best way is to prove, completely separately, that $A \Rightarrow B$ and that $B \Rightarrow A$.

Don't *abuse* the symbol $\Rightarrow$. It does not mean "so" or "hence"; do not use it just as punctuation. The symbol $\Rightarrow$ means "implies" and nothing more and nothing less.

[2] Always check things: it's a great way to learn. Also, no-one is perfect, and I'm bound to make a mistake at some point. Indeed, completely by accident, an early version of this *did* contain a mistake!

[3] Indeed, it *should* take a moment to get this!

Ideas for the tutorial

What happens in your tutorial should be negotiated by you, the other tutees, and your tutor. However, here are some ideas for discussion points. I will put up "answers" on the VLE, so if in your tutor group you don't get time to discuss all the points, you can still see some discussion.

➥ We looked at this, briefly, in lectures. Here is an example of trying to define a mathematical object which does get us into trouble. Suppose that we can make sense of $1/0$; call the resulting "number" $\diamond$. Can you deduce a contradiction? Two things to consider:

- Division is the "opposite" of multiplication, in that $(a/b) \times b = a$.
- What does multiplying by 0 give?

➥ We next consider some properties of the complex conjugate. To get some feeling for these, you could pick (at random almost) some complex number, and work out if the expression holds. But, this would not be a *proof*: for that, you need to check every possible complex number, and to do that, you need to use symbols. Anyway, we claim that the following properties hold:

- Let z be a complex number. We have that $\overline{(\overline{z})} = z$.
- Let z and w be complex numbers. We have that $|zw| = |z||w|$.

➥ Above, I gave a bad "proof" of the claim that "There are no integer solutions to $x^2 = 5$." Try to write a better proof: you should write in sentences[4], you should tell the reader what you're doing. A point for discussion is:

- Is it "obvious" that $\sqrt{5}$ is not an integer?
- Is there a proof which uses only "basic" or "simple" facts about the integer numbers?

➥ Let's think about $\Rightarrow$ a bit more. Are the following true, or not?

- $x^2 = 7 \Rightarrow x = -\sqrt{7}$
- $x > 1 \Rightarrow x > 0$
- $x < y \Rightarrow x^2 < y^2$

➥ A student attempted the following question. Do you think it is correct?

Problem: Find the solutions to $x^2 + 5x + 6 = 0$.
Solution:

$$\begin{aligned} & x^2 + 5x + 6 = 0 \\ \Rightarrow \quad & (x+3)(x+2) = 0 \\ \Rightarrow \quad & x = -3 \text{ or } x = -2. \end{aligned}$$

So the solutions are -3 and -2.

A hint: is it also true that $x^2 + 5x + 6 = 0 \Rightarrow x \neq 0$? Would this show that the solutions were "All non-zero numbers"?

[4]But don't overdo it: mathematical writing should be terse, and use rather few words. You don't need to write an essay!

Problem Set 1

Due in at the lecture on Wednesday 6 October.

You cannot expect to be able *to just do* these exercises: they will require you to puzzle things out; to write draft answers first.[5] The *tutorials* are the time to ask for help: but I don't want your tutor just to give you the answer! The tutorials will also be used to review work once you've done it: you cannot expect to get everything 100% correct on the first attempt, so it's important to learn from mistakes.

I have called this booklet a "workbook" to try to stress that the lectures and the tutors are intimately linked to the problem sets. You *need to read your notes* to do the questions: don't expect to instantly be able to do the problem set after attending the lectures.

1. (a) Rewrite each of the following complex numbers in the form $a + bi$ for some real numbers a and b. Do *not* use decimal approximations (so 0.2 or 1/5 are both okay, but while 1/3 is fine, 0.33 is not).[6]

   i. $(-3+4i)+(6+7i)$
   ii. $(9+i)+(11-i)$
   iii. $(\frac{1}{10}+\frac{2}{5}i)+(3.2-2.6i)$
   iv. $(-1-i)+(1-3i)$
   v. $(-11+23i)-(5-6i)$
   vi. $(-2-i)-(-2-i)$
   vii. $(-11.3-23i)-(2.7-23.2i)$
   viii. $7\times(23-16i)$
   ix. $(4+5i)(4-5i)$
   x. $(-1-i)(-8+i)$
   xi. $(1+i)(1+2i)(1+3i)$
   xii. $\dfrac{2+i}{3}\times\dfrac{1+2i}{5}$
   xiii. $\dfrac{1}{1+i}$
   xiv. $\dfrac{1}{i-1}$
   xv. $\dfrac{2i+4}{3-i}$
   xvi. $\dfrac{\frac{1}{2}-\frac{1}{4}i}{\frac{1}{2}-\frac{1}{3}i}$
   xvii. $\dfrac{3i-7}{i+4}$
   xviii. $(3+i)^3$
   xix. i^4
   xx. $(i+\sqrt{5})^2$
   xxi. $\left(\dfrac{1+i}{1-i}\right)^2$

   (b) Now let $z=(1+i)/\sqrt{2}$ and find the following in the form $a+bi$. Do you spot a *pattern* which would allow you to solve the last part?

   i. z^2
   ii. z^3
   iii. z^4
   iv. z^6
   v. z^8
   vi. z^{100}

   (c) Find the modulus of the following complex numbers. Again, do you see a pattern in the final three answers? (*Hint:* It will help to look above in the "Ideas for the tutorial" section.)

   i. 10
   ii. $15i$
   iii. $3+4i$
   iv. $1+i$
   v. $(1+i)^2$
   vi. $(1+i)^3$

2. For each of the following numbers, decide if they are members of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$. The final one is quite tricky!

[5] But not always! For example, if you have seen complex numbers before, then Question 1 should hopefully be easy enough to do.

[6] I also wrote this in the exam last year. If I had a pound for every student who proceeded to write "$a = 0.23$ to 2 d.p.", I would have retired by now…

(a) 5

(b) -10

(c) $15/2$

(d) $15/3$

(e) $(-5)^{-1}(-20)$

(f) $\sqrt{5}$

Let $x = \sqrt{20 + 6\sqrt{11}} - \sqrt{20 - 6\sqrt{11}}$. Is $x \in \mathbb{Z}$? (Hint: Find x^2). *Show your workings! You'll learn more if you don't use a calculator!*

Hand in your answers at the lecture. For this week only, if you miss the lecture, hand your work in to Dr Daws as soon as possible. Your tutor will mark your work and return it to you.