University of Leeds, School of Mathematics
MATH 1220 Introduction to Geometry
Details of the reduction of the equation of a conic to standard form

Recall that we want the equation of the locus of a point $P(x, y)$ whose distance from a fixed point F bears a constant ratio e to its distance from a fixed line L. We choose a coordinate system with origin at F and x-axis perpendicular to L, so that L is given by $x = -2h$ for some h. Then $PF = e\,PL$ if and only if

$$x^2 + y^2 = e^2(x + 2h)^2$$

which we can write as

$$(1 - e^2)x^2 + y^2 - 4he^2x = 4e^2h^2 \qquad (*)$$

Case (i): $e = 1$. Then (*) reads

$$y^2 = 4h(x + h)$$

and by writing $h = a$ and shifting the origin by setting $Y = y$, $X = x+h$, we obtain

$$Y^2 = 4aX\,;$$

this is the standard equation of the **parabola**.

Case (ii) and (iii): $e \neq 1$. Then (*) reads

$$(1 - e^2)\left(x^2 - \frac{4he^2}{1 - e^2}\,x\right) + y^2 = 4e^2h^2$$

Dividing by $1 - e^2$ this reads

$$\left(x^2 - \frac{4he^2}{1 - e^2}\,x\right) + \frac{y^2}{1 - e^2} = \frac{4e^2h^2}{1 - e^2}\,.$$

'Completing the square' by adding $4h^2e^4/(1 - e^2)^2$ to each side, we get

$$\left(x - \frac{2he^2}{1 - e^2}\right)^2 + \frac{y^2}{1 - e^2} = \frac{4e^2h^2}{(1 - e^2)^2}\,.$$

Case (ii): $0 < e < 1$. Then $1 - e^2 > 0$ so setting $a = 2he/(1 - e^2)$, this becomes

$$(x - ae)^2 + \frac{y^2}{1 - e^2} = a^2.$$

Writing $X = x - ae$, $Y = y$ and $b^2 = a^2(1 - e^2)$, after dividing through by a^2, this becomes

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1\,;$$

this is the standard equation of the **ellipse**.

Case (iii): $e > 1$. Then $e^2 - 1 > 0$ so setting $a = 2he/(e^2 - 1)$, this becomes

$$(x + ae)^2 - \frac{y^2}{e^2 - 1} = a^2.$$

Writing $X = x + ae$, $Y = y$ and $b^2 = a^2(e^2 - 1)$, after dividing through by a^2, this becomes

$$\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1\,;$$

this is the standard equation of the **hyperbola**.