Complex Differential Geometry

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Complex manifolds

A *complex manifold* of dimension m is a topological manifold $(M, \mathcal{U})$, such that the transition functions $\phi_U \circ \phi_V^{-1}$ are holomorphic maps between open subsets of $\mathbb{C}^m$ for every intersecting $U, V \in \mathcal{U}$.

- We have a holomorphic atlas (or "we have local complex coordinates on M.")

Remark: Obviously, a complex manifold of dimension m is a smooth (real) manifold of dimension $2m$. We will denote the underlying real manifold by $M_{\mathbb{R}}$.

Example

Complex projective space $\mathbb{C}P^m$ - the set of (complex) lines in $\mathbb{C}^{m+1}$, i.e. the set of equivalence classes of the relation

$$(z_0, \dots, z_m) \sim (\alpha z_0, \dots, \alpha z_m), \quad \forall \alpha \in \mathbb{C}^*$$

on $\mathbb{C}^{m+1} - \{0\}$.
In other words $\mathbb{C}P^m = \left(\mathbb{C}^{m+1} - \{0\}\right) / \sim$.
The complex charts are defined as for $\mathbb{R}P^m$:

$$U_i = \{[z_0, \dots, z_m]; z_i \neq 0\}, \quad i = 1, \dots, m$$

$$\phi_i : U_i \to \mathbb{C}^m, \qquad \phi_i([z_0, \dots, z_m]) = \left(\frac{z_0}{z_i}, \dots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \dots, \frac{z_m}{z_i}\right).$$

Example

1. Complex Grassmanian $\mathrm{Gr}_p(\mathbb{C}^m)$ - the set of all p-dimensional vector subspaces of $\mathbb{C}^m$.
2. The torus $T^2 \simeq S^1 \times S^1$ is a complex manifold of dimension 1.
3. As for smooth manifolds one gets plenty of examples as level sets of submersions $f : \mathbb{C}^{m+1} \to \mathbb{C}$. If f is holomorphic and df (the holomorphic differential) does not vanish at any point of $f^{-1}(c)$, then $f^{-1}(c)$ is a holomorphic manifold.For example *Fermat hypersurfaces*:

$$\left((z_0, \dots, z_m);\ \sum_{i=0}^{m} z_i^{d_i} = 1\right), \quad d_0, \dots, d_m \in \mathbb{N}.$$

4. Similarly, homogeneous f give complex submanifolds of $\mathbb{C}P^m$.
5. Complex Lie groups: $GL(n, \mathbb{C})$, $O(n, \mathbb{C})$, etc.

Almost complex manifolds

A complex manifold of (complex) dimension m is also a smooth real manifold of (real) dimension $2m$. Obviously, the converse is not true, but it turns out that there is a characterisation of complex manifolds among real ones, which is much simpler than the existence of a holomorphic atlas.

- Identifying $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ is equivalent to giving a linear map $j : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ satisfying $j^2 = -\mathrm{Id}$.
- Let $x \in M$ and let (U, ϕ_U) be a holomorphic chart around x. Define an endomorphism J_U of $T_x M_{\mathbb{R}}$ by $J_U(X) = \phi_U^{-1} \circ j \circ \phi_U(X)$.
- J_U does not depend on U, and so we have an endomorphism $J_x : T_x M_{\mathbb{R}} \to T_x M_{\mathbb{R}}$ satisfying $J_x^2 = -\mathrm{Id}$.
- The collection of all J_x, $x \in M$, defines a tensor J (of type $(1,1)$), which satisfies $J^2 = -\mathrm{Id}$ and which is called an *almost complex structure* on $M_{\mathbb{R}}$.

Definition

An *almost complex manifold* is a pair (M, J), where M is a smooth real manifold and $J : TM \to TM$ is an almost complex structure.

Thus a complex manifold is an almost complex manifold. The converse is not true, but the existence of complex coordinates follows from vanishing of another tensor.

Remark: Obviously, an almost complex manifold has an even dimension, but no every even-dimensional smooth manifold admits an almost complex structure (e.g. S^4 does not).

Remark: S^6 admits an almost complex structure, but it is still an open problem, whether it can be made into a complex manifold.

Definition

A smooth map $f : (M_1, J_1) \to (M_2, J_2)$ between two complex manifolds is called *holomorphic* if $\psi_V \circ f \circ \phi_U^{-1}$ is a holomorphic map between open subsets in $\mathbb{C}^n$, for any charts (U, ϕ_U) in M_1 and (V, ψ_V) in M_2. This is equivalent to the differential of f commuting with the complex structures, i.e. $f_* \circ J_1 = J_2 \circ f_*$.

The complexified tangent bundle

Let (M,J) be an almost complex manifold. Since J is linear, we can diagonalise it, but only after complexifying the tangent spaces. We define the *complexified tangent bundle*:

$$T^{\mathbb{C}}M = TM \otimes_{\mathbb{R}} \mathbb{C},$$

and we extend all linear endomorphisms and linear differential operators from TM to $T^{\mathbb{C}}M$ by $\mathbb{C}$-linearity.
Let $T^{1,0}M$ and $T^{0,1}M$ denote the $+i$- and the $-i$-eigenbundle of J. It is easy to verify the following:

$$T^{1,0}M = \{X - iJX;\ X \in TM\}, \quad T^{0,1}M = \{X + iJX;\ X \in TM\},$$

$$T^{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M.$$

Remark $-J$ is also an almost complex structure.
$T^{1,0}(M,-J) = T^{0,1}(M,J)$.

Holomorphic tangent vectors

Let M be a complex manifold. Recall that we have three notions of a tangent space of M at a point p: $T_p^{\mathbb{R}}M = TM$ - the real tangent space, $T_p^{\mathbb{C}}M$ - the complexified tangent space, and $T_p^{1,0}M$ (resp. $T_p^{0,1}M$) - the *holomorphic tangent space* (resp. *antiholomorphic tangent space*).
Let us choose local complex coordinates $z = (z_1, \ldots, z_n)$ near z. If we write $z_i = x_i + \sqrt{-1}y_i$, then:

$$T_pM = \mathbb{R}\left\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}\right\},$$

$$T_p^{\mathbb{C}}M = \mathbb{C}\left\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}\right\} = \mathbb{C}\left\{\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i}\right\},$$

where

$$\frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} - \sqrt{-1}\frac{\partial}{\partial y_i}\right), \quad \frac{\partial}{\partial \bar{z}_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} + \sqrt{-1}\frac{\partial}{\partial y_i}\right).$$

Consequently:

$$T_p^{1,0}M = \mathbb{C}\left\{\frac{\partial}{\partial z_i}\right\}, \quad T_p^{0,1}M = \mathbb{C}\left\{\frac{\partial}{\partial \bar{z}_i}\right\},$$

and $T_p^{1,0}M$ (resp. $T_p^{0,1}M$) is the space of derivations which vanish on anti-holomorphic functions (resp. holomorphic functions).
Now observe that:

$$V, W \in \Gamma\left(T^{1,0}M\right) \implies [V, W] \in \Gamma\left(T^{1,0}M\right),$$

and

$$V, W \in \Gamma\left(T^{0,1}M\right) \implies [V, W] \in \Gamma\left(T^{0,1}M\right).$$

The Newlander-Nirenberg Theorem

Theorem (Newlander-Nirenberg)

Let (M, J) be an almost complex manifold. The almost complex structure J comes from a holomorphic atlas if and only if

$$V, W \in \Gamma\left(T^{0,1}M\right) \implies [V, W] \in \Gamma\left(T^{0,1}M\right).$$

An almost complex structure, which comes from complex coordinates is called a *complex structure*.
Remark: An equivalent condition is the vanishing of the *Nijenhuis tensor*:

$$N(X, Y) = [JX, JY] - [X, Y] - J[X, JY] - J[JX, Y].$$

As for the proof: we just have seen the "only if" part. The "if" part is very hard. See Kobayashi & Nomizu for a proof under an additional assumption that M and J are real-analytic, and Hörmander's "Introduction to Complex Analysis in Several Variables" for a proof in full generality.

Definition

A section Z of $T^{1,0}M$ is a *holomorphic vector field* if $Z(f)$ is holomorphic for every locally defined holomorphic function f.

In local coordinates $Z = \sum_{i=1}^{n} g_i \frac{\partial}{\partial z_i}$, where all g_i are holomorphic.

Definition

A real vector field $X \in \Gamma(TM)$ is called *real holomorphic* if its $(1,0)$-component $X - iJX$ is a holomorphic vector field.

Lemma

The following conditions are equivalent:

(1) *X is real holomorphic.*

(2) *X is an infinitesimal automorphism of the complex structure J, i.e. $L_X J = 0$ (L_X - the Lie derivative).*

(3) *The flow of X consists of holomorphic transformations of M.*

The dual picture:

We can decompose the complexified cotangent bundle $\Lambda^1 M \otimes \mathbb{C}$ into

$$\Lambda^{1,0} M = \left\{\omega \in \Lambda^1 M \otimes \mathbb{C};\ \ \omega(Z) = 0\ \ \forall Z \in T^{0,1} M\right\}$$

and

$$\Lambda^{0,1} M = \left\{\omega \in \Lambda^1 M \otimes \mathbb{C};\ \ \omega(Z) = 0\ \ \forall Z \in T^{1,0} M\right\}.$$

We have

$$\Lambda^1 M \otimes \mathbb{C} = \Lambda^{1,0} M \oplus \Lambda^{0,1} M.$$

and, consequently, we can decompose the k-th exterior power of $\Lambda^1 M \otimes \mathbb{C}$ as

$$\Lambda^k M \oplus \mathbb{C} = \Lambda^k \left(\Lambda^{1,0} M \oplus \Lambda^{0,1} M\right) = \bigoplus_{p+q=k} \Lambda^p \left(\Lambda^{1,0} M\right) \otimes \Lambda^q \left(\Lambda^{0,1} M\right).$$

We write

$$\Lambda^{p,q} M = \Lambda^p \left(\Lambda^{1,0} M\right) \otimes \Lambda^q \left(\Lambda^{0,1} M\right),$$

so that

$$\Lambda^k M \oplus \mathbb{C} = \bigoplus_{p+q=k} \Lambda^{p,q} M.$$

Sections of $\Lambda^{p,q} M$ are called forms of type (p,q) and their space is denoted by $\Omega^{p,q} M$. In local coordinates, forms of type (p,q) are generated by

$$dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge d\bar{z}_{j_q}.$$

Remark: The above decomposition of $\Lambda^k M \otimes \mathbb{C}$ is valid on any almost complex manifold.
The difference between "complex" and "almost complex" lies in the behaviour of the exterior derivative d.

Theorem

Let (M,J) be an almost complex manifold of dimension $2n$. The following conditions are equivalent:

(i) *J is a complex structure.*
(ii) $d\Omega^{1,0} M \subset \Omega^{2,0} M \oplus \Omega^{1,1} M$.
(iii) $d\Omega^{p,q} M \subset \Omega^{p+1,q} M \oplus \Omega^{p,q+1} M$ *for all* $0 \le p,q \le m$.

Proof.

The only non-trivial bit is (ii) $\Longrightarrow$ (i). Use the following elementary formula for the exterior derivative of a 1-form:

$$2d\omega(Z,W) = Z\big(\omega(W)\big) - W\big(\omega(Z)\big) - \omega([Z,W]).$$

□

Using statement (iii), we decompose the exterior derivative $d : \Omega^k M \to \Omega^{k+1} M$ as $d = \partial + \bar\partial$, where $\partial : \Omega^{p,q} M \to \Omega^{p+1,q} M$, $\bar\partial : \Omega^{p,q} M \to \Omega^{p,q+1} M$.

Lemma

$$\partial^2 = 0, \quad ,\bar\partial^2 = 0, \quad \partial\bar\partial + \bar\partial\partial = 0.$$

Proof.

$0 = d^2 = (\partial + \bar\partial)^2 = \partial^2 + \bar\partial^2 + (\partial\bar\partial + \bar\partial\partial)$ and the three operators in the last term take values in different subbundles of $\Lambda^* M \otimes \mathbb{C}$. □

The Dolbeault operator

Definition

The operator $\bar\partial : \Omega^{p,q} M \to \Omega^{p,q+1} M$ is called the *Dolbeault operator*. A p-form ω of type $(p,0)$ is called holomorphic if $\bar\partial\omega = 0$.

One uses $\bar\partial$ to define *Dolbeault cohomology groups* of a complex manifold, analogously to the de Rham cohomology:

$$Z^{p,q}_{\bar\partial}(M) = \{\omega \in \Omega^{p,q}(M);\ \bar\partial\omega = 0\} \quad - \quad \bar\partial\text{-closed forms.}$$

$$H^{p,q}_{\bar\partial}(M) = \frac{Z^{p,q}_{\bar\partial}(M)}{\bar\partial\Omega^{p,q-1}(M)}.$$

A holomorphic map $f : M \to N$ between complex manifolds induces a map

$$f^* : H^{p,q}_{\bar\partial}(N) \to H^{p,q}_{\bar\partial}(M).$$

Warning: Dolbeault cohomology is not a topological invariant: it depends on the complex structure.

The ordinary Poincaré lemma says that every closed form on $\mathbb{R}^n$ is exact, and, hence, that the de Rham cohomology of $\mathbb{R}^n$ or a ball vanishes. Analogously:

Lemma (Dolbeault Lemma)

For B a ball in $\mathbb{C}^n$, $H^{p,q}_{\bar\partial}(B) = 0$, if $p+q > 0$.

For a proof, see Griffiths and Harris, p. 25.

Example (Dolbeault cohomology of $\mathbb{P}^1 = \mathbb{C}P^1$)

First of all $\Omega^{0,2}(\mathbb{P}^1) = \Omega^{2,0}(\mathbb{P}^1) = 0$, since $\dim_{\mathbb{C}} \mathbb{P}^1 = 1$. Hence $H^{0,2}_{\bar\partial}(\mathbb{P}^1) = H^{2,0}_{\bar\partial}(\mathbb{P}^1) = 0$. Also $H^{0,0}_{\bar\partial}(\mathbb{P}^1) = \mathbb{C}$.
Secondly, $\bar\partial\Omega^{1,0} = d\Omega^1$, and, hence $H^{1,1}_{\bar\partial}(\mathbb{P}^1) = \mathbb{C}$.
Thirdly, if $\omega \in Z^{1,0}_{\bar\partial}(\mathbb{P}^1) = 0$, then $d\omega = 0$ and using the vanishing of $H^1(S^2)$, we conclude that $H^{1,0}_{\bar\partial}(\mathbb{P}^1) = 0$. This also shows that there are no global holomorphic forms on $\mathbb{P}^1$, i.e. $Z^{1,0}_{\bar\partial}(\mathbb{P}^1) = 0$.
Finally, we compute $H^{0,1}_{\bar\partial}(\mathbb{P}^1) = 0$:

Complex and holomorphic vector bundles

Let M be a smooth manifold. A (smooth) *complex vector bundle* (of rank k) on M consists of a family of $\{E_x\}_{x\in M}$ of (k-dimensional) complex vector spaces parameterised by M, together with a C^∞ manifold structure on $E = \bigcup_{x\in M} E_x$, such that:

(1) The projection map $\pi : E \to M$, $\pi(E_x) = x$, is C^∞, and

(2) Any point $x_0 \in M$ has an open neighbourhood, such that there exists a diffeomorphism

$$\phi_U : \pi^{-1}(U) \longrightarrow U \times \mathbb{C}^k$$

taking the vector space E_x isomorphically onto $x \times \mathbb{C}^k$ for each $x \in U$.

The map ϕ_U is called a *trivialisation* of E over U. The vector spaces E_x are called the fibres of E. A vector bundle of rank 1 is called a *line bundle*.
Examples: The complexified tangent bundle $T^{\mathbb{C}}M$, $T^{1,0}M$, $T^{0,1}M$, bundles $\Lambda^{p,q}M$.

For any pair ϕ_U, ϕ_V of trivialisations, we have the C^∞-map $g_{UV} : U \cap V \to GL(k, \mathbb{C})$ given by

$$g_{UV}(x) = \left(\phi_U \circ \phi_V^{-1}\right)_{|\{x\}\times\mathbb{C}^k}.$$

These *transition functions* satisfy

$$g_{UV}(x) g_{VU}(x) = I, \quad g_{UV}(x) g_{VW}(x) g_{WU}(x) = I.$$

Conversely, given an open cover $\mathcal{U} = \{U_\alpha\}$ of M and C^∞-maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{C})$ satisfying these identities, there is a unique complex vector bundle $E \to M$ with transition functions $\{g_{\alpha\beta}\}$.
All operations on vector spaces induce operations on vector bundles: dual bundle E^*, direct sum $E \oplus F$, tensor product $E \otimes F$, exterior powers $\Lambda^r E$. The transition functions of these are easy to write down, e.g. if E, F are vector bundles of rank k and l with transition functions $\{g_{\alpha\beta}\}$ and $\{h_{\alpha\beta}\}$, respectively, then the transition functions of $E \oplus F$ and $E \otimes F$ are

$$\begin{pmatrix} g_{\alpha\beta}(x) & 0 \\ 0 & h_{\alpha\beta}(x) \end{pmatrix} \in GL\left(\mathbb{C}^k \oplus \mathbb{C}^l\right), \quad g_{\alpha\beta}(x) \otimes h_{\alpha\beta}(x) \in GL\left(\mathbb{C}^k \otimes \mathbb{C}^l\right).$$

An important example is the *determinant bundle* $\Lambda^k E$ of E ($k = \operatorname{rank} E$). It is a line bundle with transition functions

$$\det g_{\alpha\beta}(x) \in GL(1, \mathbb{C}) = \mathbb{C}^*.$$

A *subbundle* $F \subset E$ of a vector bundle E is a collection $\{F_x \subset E_x\}_{x \in M}$ of subspaces of the fibres E_x such that $F = \bigcup_{x\in M} F_x$ is a submanifold of E. This means that there are trivialisations of E, relative to which the transition functions look like

$$g_{UV}(x) = \begin{pmatrix} h_{UV}(x) & k_{UV}(x) \\ 0 & j_{UV}(x) \end{pmatrix}.$$

The bundle F has transition functions h_{UV}, while j_{UV} are transition functions of the quotient bundle E/F, given by E_x/F_x.
A homomorphism between vector bundles E and F on M is given by a C^∞-map $f : E \to F$, such that $f_x = f_{|E_x} : E_x \to F_x$ and f_x is linear. We have the obvious notions of $\ker(f)$ - a subbundle of E, and $\operatorname{Im}(f)$ - a subbundle of F. Also, f is an isomorphism, if each f_x is an isomorphism.
A vector bundle E on M is *trivial*, if E is isomorphic to the product bundle $M \times \mathbb{C}^k$.

Given a C^∞-map $f : M \longrightarrow N$ and a vector bundle $E \stackrel{\pi}{\longrightarrow} N$, we define the *pullback bundle* f^*E on M by

$$f^*E = \{(x,e) \in M \times E;\ f(x) = \pi(e)\},$$

i.e. $(f^*E)_x = E_{f(x)}$.
Finally, a *section* s of a vector bundle $E \longrightarrow M$ is a C^∞-map $s : M \longrightarrow E$ such that $s(x) \in E_x$ for all $x \in M$ (just like a vector field). The space of sections is denoted by $\Gamma(E)$.
Observe that trivialising a rank k bundle E over U is equivalent to giving k sections $s_1, \ldots, s_k$, which are linearly independent at every point of U. Such a collection $(s_1, \ldots, s_k)$ is called a frame for E over U.

Now, let M be a complex manifold. A *holomorphic vector bundle* $E \stackrel{\pi}{\longrightarrow} M$ is defined as a complex vector bundle, except that C^∞ is replaced by "holomorphic" everywhere.
Examples: $T^{1,0}M$, $\Lambda^{p,0}M$ (but not $\Lambda^{p,q}M$ if $q \neq 0$). The line bundle $\Lambda^{n,0}M$, $n = \dim_{\mathbb{C}} M$ is called the *canonical bundle of M*, and is denoted by K_M. Its sections are holomorphic n-forms. The dual $K_M^* = \Lambda^n\left(T^{1,0}M\right)$ is the *anti-canonical bundle*.

Example (The tautological and canonical bundles on $\mathbb{C}P^n$)

The tautological line bundle $J \longrightarrow \mathbb{C}P^n$ is defined by setting $J_A = A$, where A is a line in $\mathbb{C}^{n+1}$ defining a point of $\mathbb{C}P^n$.
We have $K_{\mathbb{C}P^n} = J^{n+1}$.

For every holomorphic bundle $E \longrightarrow M$ we define the bundles $\Lambda^{p,q}E = \Lambda^{p,q}M \otimes E$ of E-valued forms of type (p,q). The space of sections is denoted by $\Omega^{p,q}E$. If we choose a trivialisation of E, i.e. a local frame, then an element σ of $\Omega^{p,q}E$ is, in this trivialisation, $\sigma = (\omega_1, \ldots, \omega_k)$, ω_i - a local (p,q)-form on M.Now observe that we have a well-defined operator $\bar\partial : \Omega^{p,q}E \longrightarrow \Omega^{p,q+1}E$, given by:

$$\bar\partial\sigma = (\bar\partial\omega_1, \ldots, \bar\partial\omega_k)$$

in any trivialisation.
Note that $\bar\partial$ satisfies $\bar\partial^2 = 0$ and the Leibniz rule $\bar\partial(f\sigma) = \bar\partial f \otimes \sigma + f(\bar\partial\sigma)$, for any $f \in C^\infty(M)$ and $\sigma \in \Omega^{p,q}E$.
The existence of a such a natural operator $\bar\partial$ on sections of $E = \Lambda^{0,0}E$ is a remarkable property of holomorphic vector bundles. On a general complex vector bundle there is no canonical way of differentiating sections. It has to be introduced ad hoc, via the concept of *connection*:

Connections and their curvature

Definition

A connection (or *covariant derivative*) on a complex vector bundle $E \to M$ is a map $D : \Gamma(E) \to \Omega^1 E$ satisfying the Leibniz rule:

$$D(fs) = (df) \otimes s + f(Ds),$$

for any $f \in C^\infty(M)$ and $s \in \Gamma(E)$.

If we choose a frame (local basis) $(e_1, \ldots, e_k)$ for E over U, then

$$De_i = \sum_j \theta_{ij} e_j,$$

where the matrix of 1-forms is called the *connection matrix* with respect to e.

The data e and θ determine D. The connection matrix depends on the choice of e: if $(e'_1, \ldots, e'_k)$ is another frame with $e'(z) = g(z)e(z)$, $g(z) \in GL(k, \mathbb{C})$, then:

$$\theta_{e'} = g\theta_e g^{-1} + dgg^{-1}.$$

Curvature

One can extend any connection D to act on $\Omega^p E$, i.e. on E-valued p-forms, by forcing the Leibniz rule:

$$D(\omega \otimes \sigma) = d\omega \otimes \sigma + (-1)^p \omega \wedge D\sigma, \ \ \forall \omega \in \Omega^p M, \ \sigma \in \Gamma(E).$$

In particular, we have the operator $D^2 : \Gamma(E) \to \Omega^2 E$. It is tensorial, i.e. linear over $C^\infty(M)$:

$$D^2(f\sigma) = D\big(df \otimes \sigma + f(D\sigma)\big) = -df \wedge D\sigma + df \wedge D\sigma + fD^2\sigma = fD^2\sigma.$$

Consequently, D^2 is induced by a bundle map $E \to \Lambda^2 M \otimes E$, i.e. by a section of $\Lambda^2 M \otimes \mathrm{Hom}(E, E)$. This $\mathrm{Hom}(E, E)$-valued 2-form R^D is called the *curvature* of the connection D.

The curvature matrix. In a local frame, we had $De_i = \sum_j \theta_{ij} e_j$, and, hence:

$$D^2 e_i = D\left(\sum \theta_{ij} e_j\right) = \sum \left(d\theta_{ij} - \sum \theta_{ip} \wedge \theta_{pj}\right) \otimes e_j.$$

Therefore, we get the *Cartan structure equations* for the curvature matrix with respect to a frame e:

$$\Theta_e = d\theta_e - \theta_e \wedge \theta_e.$$

Hermitian metrics

Definition

Let $E \longrightarrow M$ be a complex vector bundle. A *hermitian metric* on E is a smoothly varying hermitian inner product on each fibre E_x, i.e. if $\sigma = (s_1, \dots, s_k)$ is a frame for E, then the functions $h_{ij}(x) = \langle s_i(x), s_j(x) \rangle$ are C^∞.

A complex vector bundle equipped with a hermitian metric is called a *hermitian vector bundle*.

Example: $E = T^{1,0}M$ - a hermitian metric on M.

On a hermitian vector bundle over a complex manifold, we have a canonical connection D by requiring:

- D is compatible with the metric, i.e.

$$d\langle s_1, s_2 \rangle = \langle Ds_1, s_2 \rangle + \langle s_1, Ds_2 \rangle,$$

  i.e. the metric is *parallel* for D, and
- D is compatible with the complex structure, i.e. if we write $D = D^{1,0} + D^{0,1}$ with $D^{1,0} : \Gamma(E) \longrightarrow \Omega^{1,0}E$ and $D^{0,1} : \Gamma(E) \longrightarrow \Omega^{0,1}E$, then $D^{0,1} = \bar{\partial}$.

Theorem

If $E \longrightarrow M$ is a hermitian vector bundle over a complex manifold, then there is unique connection D (called the Chern connection*) compatible with both the metric and the complex structure.*

Proof.

Let $(e_1, \dots, e_k)$ be a holomorphic frame for E and put $h_{ij} = \langle e_i, e_j \rangle$. If D is compatible with the complex structure, then De_i is of type $(1,0)$, and, as D is compatible with the metric:

$$dh_{ij} = \langle De_i, e_j \rangle + \langle e_i, De_j \rangle = \sum_p \theta_{ip} h_{pj} + \sum_p \bar{\theta}_{pj} h_{ip}.$$

The first term is of type $(1,0)$ and the second one of type $(0,1)$ and, hence:

$$\partial h_{ij} = \sum_p \theta_{ip} h_{pj}, \quad \bar{\partial} h_{ij} = \sum_p \overline{\theta_{pj}} h_{ip}.$$

Therefore $\partial h = \theta h$ and $\bar{\partial} h = h \bar{\theta}^T$ and $\theta = \partial h h^{-1}$ is the unique solution to both equations. □

Remark on the curvature of the Chern connection. If D is compatible with the complex structure, then the $(0,2)$-component of the curvature vanishes:

$$R^{0,2} = D^{0,1} \circ D^{0,1} = \bar{\partial}^2 = 0.$$

If D is also compatible with the hermitian metric, then the $(2,0)$-component vanishes as well, since

$$0 = d^2\langle e_i, e_j\rangle = \langle D^2 e_i, e_j\rangle + 2\langle De_i, De_j\rangle + \langle e_i, D^2 e_j\rangle$$

and so the $(2,0)$-component of D^2 is the negative hermitian transpose of the $(0,2)$-component.
Thus, the curvature of the Chern connection is of type $(1,1)$.

Curvature of a Chern connection in a holomorphic frame

Recall that the curvature matrix of a connection D with respect to any frame $e = (e_1, \dots, e_n)$ is given by

$$\Theta_e = d\theta_e - \theta_e \wedge \theta_e = d\theta_e - [\theta_e, \theta_e],$$

where θ_e is the connection matrix w.r.t. e.
If D is the Chern connection of a hermitian holomorphic vector bundle, and e is a holomorphic frame, then we computed: $\theta_e = \partial h h^{-1}$, where $h_{ij} = \langle e_i, e_j\rangle$. We compute $d\theta_e$:

$$(\partial+\bar{\partial})\theta_e = \bar{\partial}\theta_e + \partial(\partial h h^{-1}) = \bar{\partial}\theta_e - \partial h \wedge \partial(h^{-1}) = \bar{\partial}\theta_e + \partial h h^{-1} \wedge \partial h h^{-1}.$$

Therefore, the curvature matrix of the Chern connection w.r.t. a holomorphic frame is given by

$$\Theta = \bar{\partial}\theta.$$

In the case of a line bundle, a local non-vanishing holomorphic section s and $h = \langle s, s\rangle$:

$$\theta = \partial \log h, \quad \Theta = \bar{\partial}\partial \log h.$$

Example (Curvature of the tautological bundle of $\mathbb{C}P^n$)

J is a subbundle of the trivial bundle $\mathbb{P}^n \times \mathbb{C}^{n+1}$:

$$J = \{l \times z \in \mathbb{P}^n \times \mathbb{C}^{n+1};\ z \in l\}.$$

We have a hermitian metric on $\mathbb{P}^n \times \mathbb{C}^{n+1}$ given by the standard hermitian inner product on $\mathbb{C}^{n+1}$. It induces a hermitian metric on the line bundle J. We compute the curvature of the associated Chern connection in the patch U_0 ($z_0 \neq 0$), in which J is trivialised via $\phi : \mathbb{C}^n \times \mathbb{C} \to J$:

$$\big((z_1, \dots, z_n), u\big) \longmapsto u(1, z_1, \dots, z_n).$$

We have a non-vaninshing section $s(z_1, \dots, z_n) = (1, z_1, \dots, z_n)$, and, consequently, $h = \langle s, s\rangle = 1 + \sum_{i=1}^n |z_i|^2$. Thus, the curvature is

$$\bar\partial\partial \log h. = \sum_{i=1}^n \frac{1 + \sum_{r\neq i} |z_r|^2}{(1 + \sum_{r=1}^n |z_r|^2)^2} d\bar z_i \wedge dz_i.$$

Hermitian metrics on complex manifolds

We now look at the particular case $E = T^{1,0}M$ (M - a complex manifold). A choice of holomorphic frame is given by a choice of local holomorphic coordinates $z_1, \dots, z_n$ ($e_i = \frac{\partial}{\partial z_i}$), and a hermitian metric on $T^{1,0}M$ can be written as

$$g = \sum_{i,j} h_{ij} dz_i \otimes d\bar z_j.$$

where h_{ij} is hermitian matrix, $h^* = h$. Notice that this also defines a Hermitian metric on $T^{\mathbb{C}}M$, and by restriction a $\mathbb{C}$-valued hermitian metric on TM.
Its real part is a real-valued symmetric bilinear form, which we also denote by g, and refer to as *hermitian metric on M*. Observe that $g(JX, JY) = g(X, Y)\ \forall X, Y \in T_xM$.
The negative of the imaginary part is a skew-symmetric bilinear form ω called the *fundamental form* of g. We have $\omega(X, Y) = g(JX, Y)$, and in local coordinates:

$$g = \frac{1}{2} \mathrm{Re} \sum_{i,j} h_{ij} dz_i d\bar z_j,$$

$$\omega = \frac{\sqrt{-1}}{2} \sum_{i,j} h_{ij} dz_i \wedge d\bar{z}_j.$$

Observe that $\omega \in \Omega^{1,1} M$.

Example (Connection and curvature in dimension one)

$z = x + iy$ - a local coordinate, $\frac{\partial}{\partial z}$ - a local holomorphic frame, and a hermitian metric on $T^{1,0}M$ is written as $hdz \otimes d\bar{z}$, for a local function $h > 0$. The connection matrix of the Chern matrix is $\partial h h^{-1} = \frac{\partial \log h}{\partial z} dz$, and the curvature matrix is

$$\Theta = \bar{\partial}\partial \log h = \frac{\partial^2 \log h}{\partial \bar{z} \partial z} d\bar{z} \wedge dz = \big(-\frac{1}{4} \Delta \log h\big) dz \wedge d\bar{z}.$$

Now, the fundamental form on M is $\frac{\sqrt{-1}}{2} hdz \wedge d\bar{z}$ and hence,

$$\Theta = -\sqrt{-1} K \omega,$$

where $K = (-\Delta \log h)/2h$ is the usual Gaussian curvature of a surface.

Recall the different behaviour of positively and negatively curved surfaces. Similarly, we say that the curvature $R_E \in \Gamma\big(\Lambda^{1,1} M \otimes Hom(E, E)\big)$ of a Chern connection on a hermitian vector bundle is *positive* at $x \in M$ (notation $R_E(x) > 0$), if the hermitian matrix $R_E(x)(v, \bar{v}) \in Hom(E_x, E_x)$ is positive definite. In local coordinates, this means that $\Theta(x)(v, \bar{v})$ is positive definite $\forall v$, e.g. for surfaces with positive Gaussian curvature.
Similarly $R_E(x) \geq 0, R_E(x) \leq 0, R_E(x) < 0$. We write $R_E > 0$ etc. if the curvature is positive everywhere. Observe that the curvature of the tautological bundle $J_{\mathbb{P}^n}$ (with induced metric) is negative.
Let us compute the curvature R_F of the Chern connection of a holomorphic subbundle $F \subset E$ (with the induced hermitian metric). Let $N = F^{\perp}$. This is a C^{∞} complex subbundle of E.
If $s \in \Gamma(F)$ and $t \in \Gamma(N)$, then:

$$0 = d\langle s, t\rangle = \langle Ds, t\rangle + \langle s, Dt\rangle,$$

so that in a frame for E consisting of frames for F and N, the connection matrix of $D = D_E$ is

$$\theta_E = \begin{pmatrix} \theta_F & A \\ -A^* & \theta_N \end{pmatrix}.$$

Moreover, if the frame on F is holomorphic, then A (matrix of 1-forms) is of type $(1,0)$. Now, the curvature matrix is

$$\Theta_E = d\theta_E - \theta_E \wedge \theta_E = \begin{pmatrix} d\theta_F - \theta_F \wedge \theta_F + A \wedge A^* & * \\ * & * \end{pmatrix},$$

and so $\Theta_F = \Theta_E|_F - A \wedge A^*$. Therefore

$$R_F \leq R_E|_F,$$

so that *the curvature decreases in holomorphic subbundles*. In particular, if E is a trivial bundle with Euclidean metric (so that $R^D \equiv 0$) and $F \subset E$ is a holomorphic subbundle with the induced metric, then $R_F^D \leq 0$.

Applying this to a submanifold M of $\mathbb{C}^n$ and $F = T^{1,0}M \subset T^{1,0}\mathbb{C}^n\big|_M$ with the induced hermitian metric, we see that the curvature of $T^{1,0}M$ is always non-positive. In particular, if M is a Riemann surface, then its Gaussian curvature $K \leq 0$.

Observe that the same calculation for the quotient bundle $Q = E/F$ and conclude that

$$R_Q \geq R_E|_F,$$

i.e. the curvature increases in holomorphic quotient bundles.

As an application, consider a holomorphic vector bundle $E \to M$ "spanned by its sections", i.e. there exist holomorphic sections $s_1, \ldots, s_k \in \Gamma(E)$, such that $s_1(x), \ldots, s_k(x)$, $k \geq \operatorname{rank} E$, generate E_x for every $x \in M$. Then we have a surjective map $M \times \mathbb{C}^k \to E$, $(x, \lambda) \mapsto \sum_{i=1}^k \lambda_i s_i(x)$.

Thus, E is a quotient bundle of a trivial bundle and, if we give E the metric induced from the Euclidean metric on $M \times \mathbb{C}^k$, then $R_Q \geq 0$.

The first Chern class

We go back to a non-holomorphic setting and consider complex vector bundles over smooth manifolds. We equip such a bundle $E \longrightarrow M$ with a connection D. Recall that the curvature R^D of D is a section of $\Lambda^2 M \otimes \mathrm{Hom}(E, E)$, so basically a matrix of 2-forms. We can speak of the trace of the curvature: $\mathrm{tr}\, R^D$, which is a 2-form.
Recall the formula for the curvature matrix in a local frame: $\Theta = d\theta - [\theta, \theta]$. It follows that $\mathrm{tr}\,\Theta = \mathrm{tr}\, d\theta = d\,\mathrm{tr}\,\theta$ and, hence, $\mathrm{tr}\, R^D$ is a *closed* 2-form. It is called the *Ricci form of D*.

Lemma

The cohomology class $\left[\mathrm{tr}\, R^D\right] \in H^2_{\mathrm{DR}}(M)$ *does not depend on* D.

Proof.

$A = D - D'$ is a section of $\Lambda^1 M \otimes \mathrm{Hom}(E, E)$, so $\mathrm{tr}\, R^D - \mathrm{tr}\, R^{D'} = d\,\mathrm{tr}\, A$, which is an exact form. □

The cohomology class $c_1(E) = \frac{\sqrt{-1}}{2\pi}\left[\mathrm{tr}\, R^D\right] \in H^2_{\mathrm{DR}}(M)$ is called *the 1st Chern class* of E. It is a topological invariant.

Example

We compute $c_1(J_{\mathbb{P}^1})$. Recall that for the Chern connection induced from $\mathbb{P}^1 \times \mathbb{C}^2$, we computed the curvature matrix in the chart U_0 with holomorphic coordinate z as

$$\frac{1}{(1+|z|^2)^2} d\bar{z} \wedge dz.$$

Now, $H^2_{\mathrm{DR}}(\mathbb{P}^1)$ is identified with $\mathbb{C}$ via integration: $\omega \mapsto \int_{\mathbb{P}^1} \omega$. Thus

$$c_1(J_{\mathbb{P}^1}) = \frac{\sqrt{-1}}{2\pi} \int_{\mathbb{C}} \frac{1}{(1+|z|^2)^2} d\bar{z} \wedge dz = \frac{1}{2\pi} \int_{[0,2\pi]\times[0,\infty)} \frac{r}{(1+r^2)^2} d\theta \wedge dr,$$

where $z = re^{i\theta}$. Taking the orientation into account we obtain

$$c_1(J_{\mathbb{P}^1}) = \frac{-1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} \frac{r}{(1+r^2)^2} d\theta dr = -1.$$

It follows from this that $c_1(J_{\mathbb{P}^n}) = -1$ for any n.

Let E and F be bundles of ranks m and n, respectively. Then:

(i) $c_1(\Lambda^m E) = c_1(E)$,

(ii) $c_1(E \oplus F) = c_1(E) + c_1(F)$,

(iii) $c_1(E \otimes F) = nc_1(E) + mc_1(F)$,

(iv) $c_1(E^*) = -c_1(E)$,

(v) $c_1(f^* E) = f^* c_1(E)$.

Let M be a complex manifold. The first Chern class $c_1(M)$ of M is $c_1\left(T^{1,0}M\right) = c_1(K^*)$, i.e. the first Chern class of the anti-canonical bundle of M.

Example

$$c_1(\mathbb{P}^n) = c_1(K^*) = c_1\left(\left(J^*\right)^{\otimes(n+1)}\right) = (n+1)c_1(J^*) = n+1.$$

For $n = 1$, we get $c_1(\mathbb{P}^2) = 2$.

This is just the Gauss-Bonnet theorem: for any hermitian metric on a compact surface S :

$$c_1(S) = \frac{1}{2\pi}\int_S K\omega = \chi(S).$$

Examples of manifolds with $c_1(M) = 0$:

Observe that $c_1(M) = 0$, if the canonical bundle K_M is trivial, i.e. there exists a non-vanishing holomorphic n-form on M ($n = \dim_{\mathbb{C}} M$).

- $\mathbb{C}^n$. Other boring examples include any M with $H^2(M) = 0$.
- Quotients of $\mathbb{C}^n$ by finite groups of holomorphic isometries, e.g by lattices: abelian varieties (complex tori).
- The quadric $Q = \{(z_1, z_2, z_3) \in \mathbb{C}^3;\ z_1^2 + z_2^2 + z_3^2 = 1\}$. This is a complexification of S^2 and so $H^2(Q) \neq 0$. The following holomorphic 2-form is non-vanishing on Q and trivialises K_Q:

$$z_1 dz_2 \wedge dz_3 + z_2 dz_3 \wedge dz_1 + z_3 dz_1 \wedge dz_2.$$

- The famous K3-surface (one of them, anyway):
  $S = \{[z_0, z_1, z_2, z_3] \in \mathbb{P}^3;\ z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\}$.

We'll show how to compute c_1 of hypersurfaces. Let $V \subset M$ be a complex submanifold of dimension 1. We have the exact sequence of holomorphic vector bundles over V:
$0 \to N_V^* \to \Lambda^{1,0}M_{|V} \to \Lambda^{1,0}V \to 0$.
Taking the maximal exterior power gives: $K_{M|V} \simeq K_V \otimes N_V^*$, so

$$K_V = K_{M|V} \otimes N_V.$$

Now, observe that, if $f = 0$ is a local (say, on $U \subset V$) defining equation for V, then, by definition, df generates N_V^* over U.
If, on $U' \subset V$, the equation of V is $f' = 0$, then on $U \cap U'$:

$$df = d\Big(\frac{f}{f'}f'\Big) = d\Big(\frac{f}{f'}\Big)f' + \frac{f}{f'}df' = \frac{f}{f'}df',$$

so the transition functions for N_V^* are f/f'.
We can now easily finish the computation for K3: $N_S^* \simeq K_{\mathbb{P}^3|S}$.

Higher Chern classes

The first Chern class of a complex vector bundle was defined using the trace of the curvature $R^D \in \Gamma\big(\Lambda^2 M \otimes \mathrm{Hom}(E,E)$. We can use other invariant polynomials defined on matrices, e.g. the determinant or the sum of the squares of eigenvalues.
For a $k \times k$ matrix M, let us write

$$\det(t + M) = \sum_{i=0}^{k} P_i(M) t^{k-i}.$$

so that $P_1(M) = \operatorname{tr} M$, $P_k(M) = \det M$, etc. Each P_i is a homogeneous polynomial of degree k in the entries of A.
We can allow matrices over any commutative algebra - the $P_i(M)$ are well defined - in particular over *the even part of the exterior algebra of a smooth manifold*. Applying this to R^D or its curvature matrix, we obtain closed forms $P_i(R^D) = P_i(\Theta) \in \Omega^{2i}M$.

Definition

The i-th Chern class of a complex vector bundle E over a smooth manifold M is the cohomology class

$$c_i(E) = \left[P_i\left(\frac{\sqrt{-1}}{2\pi} R^D \right) \right] \in H^{2i}_{\mathrm{DR}}(M), \quad i = 1, \ldots, k.$$

Once again, it does not depend on the connection D (exercise). Also, for a complex manifold M, we define the Chern classes of M to be the Chern classes of $T^{1,0}M$.

The Gauss-Bonnet theorem generalise to higher dimensions as follows: if M is a compact complex manifold of dimension n, then

$$c_n(M) = \chi(M).$$

There are of course purely topological ways of defining Chern classes!

Chern classes of a holomorphic bundles

Now E is hermitian holomorphic vector bundle over a complex manifold M. Recall that the curvature of a Chern connection D is of type $(1,1)$ and that its matrix is skew-hermitian, so that $P_i(\frac{\sqrt{-1}}{2\pi} R^D)$ is a real (i,i)-form, and consequently, for any holomorphic bundle

$$c_i(E) \in H^{i,i}(M) \cap H^{2i}_{\mathrm{DR}}(M; \mathbb{R}).$$

Recall also that we defined notions of the curvature being $> 0, < 0, \geq 0, \leq 0$. For example, nonnegativity meant that $R^D(v, \bar{v})$ has all eigenvalues nonnegative, for any $v \in T^{1,0}M$. This means that

$$P_i(R^D)(v_1, \bar{v}_1, \ldots, v_i, \bar{v}_i)) \geq 0 \quad \forall v_1, \ldots, v_i \in T^{1,0}M,$$

and consequently that

$$\int_Z c_i(E) \geq 0$$

for any i-dimensional complex submanifold Z of M. In particular, this holds for any holomorphic vector bundle generated by its sections. Note that these conditions depend only on $c_i(E)$, not on the connection or its curvature.

Prescribing the Ricci curvature of Chern connections

Thus, we say that $c_1(E) \geq 0$ (resp. $> 0, < 0, \leq 0$), if the cohomology class $c_1(E) \in H^{1,1}(M)$ can be locally represented by a closed real 2-form ϕ, such that in local complex coordinates

$$\phi = \frac{\sqrt{-1}}{2\pi} \sum \phi_{\alpha\bar{\beta}} dz_\alpha \wedge d\bar{z}_\beta,$$

and the hermitian matrix $[\phi_{\alpha\bar{\beta}}]$ has all eigenvalues nonnegative (resp. positive, negative, nonpositive).
E.g., any holomorphic vector bundle generated by its sections satisfies $c_1(E) \geq 0$.
Let now a closed real $(1,1)$-form ϕ represent $c_1(E)$. We ask

Is there a connection D on E such that $\operatorname{tr} R^D = -2\pi\sqrt{-1}\phi$?

For example, suppose that $c_1(E) = 0$. It is clearly represented by $\phi \equiv 0$ and we would like to know whether there is a hermitian metric, such that the associated Chern connection is Ricci-flat (i.e. $\operatorname{tr} R^D \equiv 0$).

Let $\langle , \rangle$ be a hermitian metric on E. Recall that in a local holomorphic frame $(e_1, \ldots, e_n)$ with the associated matrix $h_{ij} = \langle e_i, e_j \rangle$, we had the following formula for R^D:

$$\Theta = \bar{\partial}(\partial h h^{-1}).$$

Therefore, the Ricci form $\operatorname{tr} R^D$ is represented in this local frame by

$$\bar{\partial}\partial(\log \det h).$$

Now we modify $\langle , \rangle$ by taking $e^{f/k}\langle , \rangle$, where $k = \operatorname{rank} E$ and f is a smooth real function on M. Let $h'_{ij} = e^{f/k}\langle e_i, e_j \rangle$. Then $\det h' = e^f \det h$ and the Ricci forms of the two Chern connections are related by

$$\operatorname{tr} R^{D'} - \operatorname{tr} R^D = \bar{\partial}\partial f.$$

Therefore we find a hermitian metric with $\operatorname{tr} R^{D'} = \phi$, providing we can solve the equation

$$\bar{\partial}\partial f = \phi - \operatorname{tr} R^D.$$

The right-hand side is a closed imaginary $(1,1)$-form cohomologous to 0: $\left[\phi - \operatorname{tr} R^D\right] = -2\pi\sqrt{-1}(c_1(E) - c_1(E))$.

Therefore the answer to our question: can we find a hermitian metric on E, the Ricci form of which is a given form ϕ, is **yes** on manifolds M for which the following condition (called *the global* $\partial\bar\partial$*-lemma*) holds:

- *Any exact real* $(1,1)$*-form* β *on* M *is of the form* $\sqrt{-1}\bar\partial\partial f$*, for a smooth function* $f : M \to \mathbb{R}$.

Lemma

Let M *be a complex manifold with* $H^{0,1}(M) = 0$*. Then the global* $\partial\bar\partial$*-lemma holds on* M.

Proof. Since β is exact, there exists a real 1-form α such that $d\alpha = \beta$. We decompose α as $\tau + \tau'$, where τ is $(1,0)$ and τ' is $(0,1)$. We must have $\tau' = \bar\tau$.
Now, $\beta = (\partial + \bar\partial)(\tau + \bar\tau) = \partial\tau + (\bar\partial\tau + \partial\bar\tau) + \bar\partial\bar\tau$. Comparing the types, we get:

$$\beta = \bar\partial\tau + \partial\bar\tau, \quad \partial\tau = 0, \quad \bar\partial\bar\tau = 0.$$

Since $H^{0,1}(M) = 0$, there exists a function $u : M \to \mathbb{C}$ such that $\bar\tau = \bar\partial u$. Therefore $\tau = \partial\bar u$, and
$\beta = \bar\partial\tau + \partial\bar\tau = \bar\partial\partial\bar u + \partial\bar\partial u = \partial\bar\partial(u - \bar u) = 2i\partial\bar\partial(\operatorname{Im} u)$.

The result follows with $f = 2\operatorname{Im} u$. □

Kähler metrics

We now consider the case $E = TM$, where M is a complex manifold with a complex structure J. TM is a complex vector bundle - J allows us to identify T_xM with $\mathbb{C}^n$, for every n. In other words we identify

$$TM \simeq T^{1,0}M, \quad X \longmapsto X - iJX.$$

Recall that a hermitian metric on M is a smoothly varying inner product g on each T_xM, such that $g(JX, JY) = g(X,Y)$ for all $X, Y \in T_xM$.
We will now consider connections on TM. First, of all observe, that for such a connection $D : \Gamma(TM) \to \Omega^1(M) \otimes \Gamma(TM)$, the (covariant) directional derivative $D_Z(X) := D(X)(Z)$ of a vector field is again a vector field.
We now reinterpret the two properties defining the Chern connection D of g:
1. D being metric, i.e. $dg(X,Y) = g(DX,Y) + g(X,DY)$, is equivalent to

$$Z.g(X,Y) = g(D_ZX, Y) + g(X, D_ZY) \quad \forall X, Y, Z \in \Gamma(TM).$$

2. D being compatible with the complex structure, i.e. $D^{0,1} = \bar{\partial}$, is equivalent to

$$DJ = 0, \text{ i.e. } D_Z(JX) = JD_Z(X) \quad \forall X, Z \in \Gamma(TM).$$

i.e. J is *parallel* for D (follows from the fact that the connection matrix in the holomorphic frame has type $(1,0)$).
On the other hand, since g is a Riemannian metric, i.e. a smoothly varying inner product, there exists a unique connection (the *Levi-Civita connection*) ∇ on TM, which is again metric and it has zero *torsion*, i.e.

$$\nabla_X Y - \nabla_Y X = [X, Y], \quad \forall X, Y \in \Gamma(TM).$$

Thus, on a complex manifold with a hermitian metric, we have two natural connections: Chern and Levi-Civita. We ask for which hermitians metrics the two coincide.

Theorem

Let g be a hermitian metric on a complex manifold (M, J). The following conditions are equivalent:

(i) *J is parallel for the Levi-Civita connection ∇.*
(ii) *The Chern connection D has zero torsion.*
(iii) *The Levi-Civita and the Chern connections coincide.*
(iv) *The fundamental form ω of g is closed, $d\omega = 0$, (recall that $\omega(X, Y) = g(JX, Y)$).*
(v) *For each point $p \in M$, there exists a smooth real function in a neighbourhood of p, such that $\omega = i\partial\bar{\partial}f$.*
(vi) *For each point $p \in M$, there exist holomorphic coordinates w centred at p, such that $g(w) = 1 + O(|w|^2)$.*

Proof. (i), (ii), and (iii) are clearly equivalent from the uniqueness properties. We show $(i) \implies (iv) \implies (v) \implies (vi) \implies (i)$.

$(i) \implies (iv)$. Since $\nabla g = 0$ and $\nabla J = 0$, $\nabla\omega = 0$. For any p-form and any torsion-free connection ∇ one has (exercise):

$$d\omega(X_0,\dots,X_p)=\sum_{i=0}^{p}(-1)^i\big(\nabla_{X_i}\omega\big)\big(X_0,\dots,\widehat{X}_i,\dots,X_p\big),$$

which implies that parallel forms are closed.

$(iv) \implies (v)$. This is the $\partial\bar{\partial}$-lemma applied to a ball B in $\mathbb{C}^n$ (Dolbeault Lemma says that $H^{0,1}(B)=0$).

$(v) \implies (vi)$. We can choose local coordinates around p_0, so that $\omega = i\sum_{l,m}\omega_{lm}dz_l\wedge d\bar{z}_m$, where

$$\omega_{lm}=\frac{1}{2}\delta_{lm}+\sum_j(a_{jlm}z_j+b_{jlm}\bar{z}_j)+O(|z|^2).$$

Reality implies that $a_{jlm}=\overline{b_{jml}}$. On the other hand, using (v), we have

$$a_{jlm}=\frac{\partial^3 f}{\partial z_j\partial z_l\partial\bar{z}_m},$$

so $a_{jlm}=a_{ljm}$. We now put

$$w_m=z_m+\sum_{j,l}a_{jlm}z_jz_l, \quad \implies \quad dw_m=dz_m+2\sum_{j,l}a_{jlm}z_jdz_l,$$

and compute:

$$\begin{aligned}\frac{1}{2}i\sum_m dw_m\wedge d\bar{w}_m &= \frac{1}{2}i\sum_m dz_m\wedge d\bar{z}_m + i\sum_{j,l,m}a_{jlm}z_jdz_l\wedge d\bar{z}_m\\ &\quad + i\sum_{j,l,m}\overline{a_{jlm}}\bar{z}_j dz_m\wedge d\bar{z}_l+O(|z|^2)\\ &= i\sum_{l,m}\omega_{lm}dz_l\wedge d\bar{z}_m+O(|z|^2)=\omega+O(|z|^2)=\omega+O(|w|^2).\end{aligned}$$

$(vi) \implies (i)$ If we write $z_i=x_i+\sqrt{-1}y_i$ for the coordinates obtained in (v), then the connection matrix of the Levi-Civita connection (*Christoffel symbols)* is equal to zero at p_0. Consequently, ∇J vanishes at p_0, and, since p_0 is arbitrary, it vanishes everywhere. □

A hermitian metric on a complex manifold satisfying the equivalent conditions $(i)-(vi)$ is called a *Kähler metric*.

The fundamental form ω of a Kähler metric is called a Kähler form and the local function f in (v) is called a local *Kähler potential*.

Examples:
1. The standard Euclidean metric on $\mathbb{C}^n$.

$$g=\frac{1}{2}\operatorname{Re}\sum_{s=1}^{n} dz_s\otimes d\bar{z}_s,\quad \omega=\frac{i}{2}\operatorname{Re}\sum_{s=1}^{n} dz_s\wedge d\bar{z}_s=\frac{i}{2}\partial\bar{\partial}|z|^2.$$

Thus $f(z)=\frac{1}{2}|z|^2$ is a global Kähler potential on $\mathbb{C}^n$. The pair (g,J) is $U(n)$-invariant.

2. The Fubini-Study metric on $\mathbb{C}P^n$.
For a $z=(z_0,\dots,z_n)\in\mathbb{C}^{n+1}-\{0\}$, we put

$$\omega=i\partial\bar{\partial}\log(|z|^2).$$

This is invariant under rescaling by $\mathbb{C}^*$ and, so, it induced a real closed $(1,1)$-form on $\mathbb{P}^n$. We need to check that the corresponding symmetric form $g(X,Y)=\omega(X,JY)$ is positive-definite. We compute first ω in the chart $U_0=\{z_0\neq 0\}$, where the local coordinates are $w_i=\frac{z_s}{z_0}$, $s=1,\dots,n$:

$$\omega=i\partial\bar{\partial}\log(1+|w|^2)=\partial\left(\frac{i}{1+|w|^2}\sum_{s=1}^{n} w_s d\bar{w}_s\right)=$$

$$\frac{i}{1+|w|^2}\sum_{s=1}^{n} dw_s\wedge d\bar{w}_s-\frac{i}{(1+|w|^2)^2}\left(\sum_{s=1}^{n}\bar{w}_s dw_s\right)\wedge\left(\sum_{s=1}^{n} w_s d\bar{w}_s\right)$$

Observe that ω is $U(n+1)$- invariant, and so it is enough to check positivity at any point $p\in\mathbb{P}^n$, e.g $p=[1,0,\dots,0]$, i.e. $w=0$. OK!

Remark. The Fubini-Study metric is the quotient metric on $\mathbb{C}P^n \simeq S^{n+1}/S^1$ obtained from the standard round metric on S^{n+1}.

These two basic examples ($\mathbb{C}^n$ and $\mathbb{C}P^n$) yield plenty of others, because:

Lemma

A complex submanifold of a Kähler manifold, equipped with the induced metric, is Kähler.

Proof. Let $(N, J, g_N) \subset (M, J, g_M)$ as in the statement. The fundamental form ω_N of g_N is just the pullback (restriction) of the fundamental form ω_M of g_M, hence it is closed. $\square$
Observe also that a product of two Kähler manifolds is Kähler.

On the other hand, many complex manifolds do not admit any Kähler metric, because:

Lemma

If M is a compact Kähler manifold, then $H^{2q}_{\mathrm{DR}}(M) \neq 0$, $q \leq n = \dim_{\mathbb{C}} M$.

Proof.

Let ω be the Kähler form. Then ω^q is a closed form, which is not exact. Indeed, had we $\omega^q = d\psi$, then $\omega^n = d(\psi \wedge \omega^{n-q})$, and, so:

$$\mathrm{vol}(M) = \int_M \omega^n = \int_M d(\psi \wedge \omega^{n-q}) = 0,$$

which is impossible. $\square$
Thus, for example, there is no Kähler metric on $S^1 \times S^{2n-1}$ for $n \geq 2$ (this is a complex manifold - see Examples Set 1).
Another easily provable topological restriction is

Lemma

Let M be a compact Kähler manifold. Then the identity map on $\Omega^q M$ induces an injective map

$$H^{q,0}_{\bar\partial}(M) \hookrightarrow H^q_{\mathrm{DR}}(M, \mathbb{C}),$$

i.e. every holomorphic $(q,0)$-form is closed and never exact.

Proof. Let η be a holomorphic $(q,0)$-form. Write η in a local unitary frame $\{\phi_i\}$ as $\eta = \sum f_I \phi_I$ (I runs over subsets of cardinality q and ϕ_I is the wedge product of ϕ_j with $j \in I$). Then $\eta \wedge \bar{\eta} = \sum f_I \bar{f}_J \phi_I \wedge \bar{\phi}_J$.
On the other hand, $\omega = \frac{\sqrt{-1}}{2} \sum \phi_i \wedge \bar{\phi}_i$, and ($n = \dim_{\mathbb{C}} M$):

$$\omega^{n-q} = c_q \sum_{\#K=n-q} \phi_K \wedge \bar{\phi}_K.$$

Hence

$$\eta \wedge \bar{\eta} \wedge \omega^{n-q} = c'_q \sum_{\#I=q} |f_I|^2 \omega^n,$$

since the only non-zero wedge-products arise for K disjoint from I and J (so for $I = J$). In particular, if $\eta \neq 0$, then the integral over M of the LHS is non-zero. If, however, $\eta = d\psi$, then :

$$\eta \wedge \bar{\eta} \wedge \omega^{n-q} = d\left(\psi \wedge \bar{\eta} \wedge \omega^{n-q}\right),$$

since $d\bar{\eta} = 0$ and $d\omega = 0$, and we obtain a contradiction.
Thus, a non-zero holomorphic $(q,0)$-form is never exact. To show that η is closed, note: $d\eta = (\partial + \bar{\partial})\eta = \partial\eta$, and, so, $d\eta$ is a holomorphic $(q+1,0)$-form, hence $d\eta = 0$. □

The last result is a particular case of a much stronger fact, which goes under the name of *Hodge relations* or *Hodge decomposition*:

Theorem (Hodge)

The complex cohomology of a compact Kähler manifold satisfies:

$$H^r(M,\mathbb{C}) = \bigoplus_{p+q=r} H^{p,q}(M), \qquad H^{p,q}(M) = \overline{H^{q,p}(M)}.$$

In particular: *for r odd,* $\dim H^r(M)$ *is even.*

To give even an idea of a proof of this theorem of Hodge requires a substantial detour.

Before, let us see how badly this theorem fails for non-compact Kähler manifolds:

Example

$M = \mathbb{C}^2 - \{0\}$, so that $H^1_{\mathrm{DR}}(M) = 0$. We shall show that $\dim H^{0,1}_{\bar\partial}(M) = +\infty$. Let $U_1 = \{z_1 \neq 0\} = \mathbb{C}^* \times \mathbb{C}$, $U_2 = \{z_2 \neq 0\} = \mathbb{C} \times \mathbb{C}^*$, so that $U_1 \cup U_2 = M$, $U_1 \cap U_2 = \mathbb{C}^* \times \mathbb{C}^*$.
Let $\lambda_1, \lambda_2 : M \to \mathbb{R}$ be a partition of unity subordinated to U_1, U_2 and let f be a holomorphic function on $U_1 \cap U_2$. Then $g_1 = \lambda_2 f$ defines a smooth function on U_1 and $g_2 = -\lambda_1 f$ defines a smooth function on U_2. Observe that on $U_1 \cap U_2$, $\bar\partial(g_1 - g_2) = \bar\partial f = 0$, and so we can define a $(0,1)$-form ω on M, with $\bar\partial\omega = 0$, by

$$\omega = \begin{cases} \bar\partial g_1 = f\bar\partial\lambda_2 & \text{on } U_1 \\ -\bar\partial g_2 = -f\bar\partial\lambda_1 & \text{on } U_2. \end{cases}$$

Suppose that $\omega = \bar\partial h$ for some $h \in C^\infty(M)$. Then $\bar\partial(g_1 - h) = 0$ on U_1 and $\bar\partial(g_2 - h) = 0$, and, so $g_1 - h$ is holomorphic on U_1, $g_2 - h$ is holomorphic on U_2. Then $f = g_1 - g_2 = (g_1 - h) - (g_2 - h)$, and, hence, $f = u_1 + u_2$, where u_i is holomorphic on U_i, $i = 1, 2$.
But any convergent Laurent series f on $U_1 \cap U_2$, which contains $z_1^m z_2^n$, $m, n < 0$, is not $u_1 + u_2$, so ω defined by such an f is not exact.

An idea of a proof of the Hodge theorem

Let V be a vector space, equipped with an inner product $\langle , \rangle$. Then there is an induced inner product on the tensor powers $V^{\otimes k}$. By restriction, there is an inner product on the exterior powers $\Lambda^k V$. It can be described by choosing an orthonormal basis $(e_1, \dots, e_n)$ and decreeing the following basis of $\Lambda^k V$:

$$\{e_{i_1} \wedge \cdots \wedge e_{i_k};\ \ 1 \leq i_1 < \cdots < i_k \leq n\}$$

to be orthonormal. Now, if (M, g) is an oriented Riemannian manifold, then we can do the above on each tangent space $T_x M$. Since a dual of a vector space with an inner product also has an inner product, we get an inner product on $\Lambda^k T^*_x M$, and, if (M, g) is *oriented*, i.e. we have a nonvanishing volume form dV, *and compact*, then we can define an inner product on differential forms in $\Omega^k M$:

$$\langle \alpha, \beta \rangle = \int_M \langle \alpha_{|x}, \beta_{|x} \rangle dV.$$

The idea is that in every cohomology class in $H^k_{\mathrm{DR}}(M)$ we seek a representative α with the smallest norm.

We also show that such a representative must be unique (an element with the smallest norm in a closed affine subspace of a Hilbert space). We do the same for the Dolbeault cohomology classes $H^{p,q}_{\bar\partial}(M)$. Finally, we show that, if M is Kähler, then, if we decompose the $\alpha \in H^k_{\mathrm{DR}}(M,\mathbb{C})$ with minimal norm into forms of type (p,q), $p+q=k$, then each component has the minimal norm in $H^{p,q}_{\bar\partial}(M)$.

How to find the element of smallest norm in a cohomology class $[\psi] \in H^k_{\mathrm{DR}}(M)$?

We are looking for the element of smallest norm in an affine subspace $P = \{\psi + d\eta;\ \ \eta \in \Omega^{k-1}M\}$. For M compact, $\Omega^k(M)$ with the inner product $\langle , \rangle$ is a pre-Hilbert space (essentially an L^2-space), and were P a *closed* subspace, we could find an element of the smallest norm by the orthogonal projection ($\Omega^k(M) = d\Omega^{k-1}M \oplus (d\Omega^{k-1}M)^{\perp}$). The orthogonal projection, on the other hand, can be expressed via the adjoint operator to d:

$$\|\psi + d\eta\|^2 = \|\psi\|^2 + \|d\eta\|^2 + 2\langle \psi, d\eta\rangle = \|\psi\|^2 + \|d\eta\|^2 + 2\langle d^*\psi, \eta\rangle.$$

Thus, if $d^*\psi = 0$, then ψ has the smallest norm in P.

Hodge dual

So cohomology classes should be represented by forms ψ such that $d\psi = 0$ and $d^*\psi = 0$, once we define $d^* : \Omega^{k+1} \to \Omega^k$.

To define d^*, we go back to the situation described earlier: V is a vector space, $\dim V = n$, equipped with an inner product $\langle , \rangle$ and an orientation, i.e. a preferred element of $\Lambda^n V$, e.g. $e_1 \wedge \cdots \wedge e_n$. Any $\alpha \in \Lambda^{n-k} V$ defines a linear functional ϕ on $\Lambda^k V$ by:

$$\omega \wedge \alpha = \phi(\omega) e_1 \wedge \cdots \wedge e_n.$$

Such a functional must be represented by $\omega \mapsto \langle \omega, \tau\rangle$ for some $\tau \in \Lambda^k V$. This way, we get a 1-1 correspondence between $\Lambda^k V$ and $\Lambda^{n-k} V$:

$$* : \Lambda^k V \to \Lambda^{n-k} V, \quad \tau \mapsto \alpha,$$

$$\omega \wedge *\tau = \langle \omega, \tau\rangle e_1 \wedge \cdots \wedge e_n.$$

The operator $*$ is called the *Hodge dual*. Notice, in particular:

$$*1 = e_1 \wedge \cdots \wedge e_n, \quad \langle *\tau_1, *\tau_2\rangle = \langle \tau_1, \tau_2\rangle, \quad *^2 = (-1)^{k(n-k)} \text{ on } \Lambda^k V.$$

Again, we obtain such a $*$ on differential forms of an oriented Riemannian manifold (M, g).

Example: Take $M = \mathbb{R}^3$ with the Euclidean metric $g = \sum_{i=1}^{3} dx_i \otimes dx_i$ and the standard orientation $dx_1 \wedge dx_2 \wedge dx_3$. We get:

$$*dx_1 = dx_2 \wedge dx_3, \quad *dx_2 = dx_3 \wedge dx_1, \quad *dx_3 = dx_1 \wedge dx_2.$$

Now, observe that $*d* : \Omega^{k+1} \to \Omega^k$ and we have (dV denotes the volume form determining orientation):

$$\langle d\alpha, \beta \rangle dV = d\alpha \wedge *\beta = d(\alpha \wedge *\beta) - (-1)^k \alpha \wedge d * \beta, \quad \text{so}$$

$$\langle d\alpha, \beta \rangle dV - d(\alpha \wedge *\beta) = -(-1)^k \alpha \wedge d * \beta$$

$$= -(-1)^k (-1)^{k(n-k)} \alpha \wedge *^2 d * \beta = -(-1)^{kn} \langle \alpha, *d * \beta \rangle dV.$$

Therefore, the operator $d^* = (-1)^{kn+1} * d* : \Omega^{k+1} \to \Omega^k$ is the adjoint of d, and is called the *codifferential*. A form ω, such that $d^*\omega = 0$ is called *co-closed*.

We need to show that on a compact oriented manifold (M, g), any cohomology class has a representative ψ with $d^*\psi = 0$ (and $d\psi = 0$).

The Riemannian Laplacian

Observe that such a form also satisfies $(dd^* + d^*d)\psi = 0$. The operator

$$\Delta = dd^* + d^*d : \Omega^k M \to \Omega^k M$$

is called the *Riemannian Laplacian* or the *Laplace-Beltrami operator*.

Let's check that we really get the usual Laplacian on functions on $\mathbb{R}^n$:

$$(dd^* + d^*d)f = d^*df = - * d * \left(\sum \frac{\partial f}{\partial x_j} dx_j \right) = - * d \left(\sum \frac{\partial f}{\partial x_j} * dx_j \right)$$

$$= - * \left(\sum \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i \wedge * dx_j \right) = - * \left(\sum \frac{\partial^2 f}{\partial x_i^2} \right) dV = \Delta f.$$

In general, let a Riemannian metric be given in local coordinates by $g = \sum_{ij} dx_i \otimes dx_j$. Let $[g^{ij}]$ be the inverse matrix of $[g_{ij}]$ (this defines the metric on T^*M), and $|g| = \det[g_{ij}]$. Then

$$\Delta_g f = -\frac{1}{\sqrt{|g|}} \sum_{i,j} \frac{\partial}{\partial x_i} \left(\sqrt{|g|} g^{ij} \frac{\partial f}{\partial x_j} \right).$$

A form ψ such that $\Delta\psi = 0$ is called *harmonic*. Thus, a form satisfying $d\psi = d^*\psi = 0$ is harmonic. On a compact manifold, we also have the converse:

Lemma

If M is compact, then any harmonic form ψ satisfies $d\psi = d^*\psi = 0$.

Proof. "Integration by parts":

$$0 = \int_M \langle \Delta\psi, \psi \rangle dV = \int_M \langle dd^*\psi + d^*d\psi, \psi \rangle dV = \int_M \left(|d^*\psi|^2 + |d\psi|^2 \right) dV.$$

Corollary

A harmonic function on a compact Riemannian manifold is constant.

Let $\mathcal{H}^k(M)$ be the vector space of harmonic k-forms, i.e.

$$\mathcal{H}^k(M) = \{\psi \in \Omega^k M;\ \ \Delta\psi = 0\}.$$

For M compact and oriented, let $\langle , \rangle$ denote the global inner product on $\Omega^k M$ (given by integration).

Theorem (Hodge-de Rham)

On a compact oriented manifold (M, g):

$$\Omega^k M = \mathcal{H}^k(M) \oplus d\Omega^{k-1} M \oplus d^*\Omega^{k+1} M,$$

where the summands are orthogonal with respect to the global inner product $\langle , \rangle$.

Before discussing the proof, let's see some applications:

Corollary (Hodge isomorphism)

The natural map $f : \mathcal{H}^k(M) \to H^k_{\mathrm{DR}}(M)$, given by $\psi \mapsto [\psi]$ is an isomorphism.

Proof. We known that $d\psi = 0$, so the map is well defined. Since $\mathcal{H}$ is orthogonal to exact forms, the kernel of f is zero. Finally, let $[\omega] \in H^k_{\mathrm{DR}}(M)$ and decompose $\omega = \omega^H + d\lambda + d^*\phi$ with ω^H harmonic. Then

$$0 = \langle d\omega, \phi \rangle = \langle dd^*\phi, \phi \rangle = \langle d^*\phi, d^*\phi \rangle.$$

Therefore $d^*\phi = 0$ and $[\omega] = [\omega^H + d\lambda] = \omega^H$, so f is surjective. $\square$

Corollary (Poincaré duality)

On a compact oriented n-dimensional manifold M,
$H^k_{\mathrm{DR}}(M) \simeq H^{n-k}_{\mathrm{DR}}(M)$.

Proof. Put any metric on M. The Hodge dual $*$ gives an isomorphism $\mathcal{H}^k(M) \simeq \mathcal{H}^{n-k}(M)$. □

On the proof of the Hodge-de Rham Theorem

It is clear that the three summands $\mathcal{H}^k(M)$, $d\Omega^{k-1}M$, $d^*\Omega^{k+1}M$ are mutually orthogonal: if ω is harmonic, then $\langle\omega, d\phi\rangle = \langle d^*\omega, \phi\rangle = 0$. Similarly $\langle\omega, d^*\psi\rangle = 0$, and, finally:

$$\langle d\phi, d^*\psi\rangle = \langle dd\phi, \psi\rangle = 0.$$

The hard part is to show that their sum is all of $\Omega^k M$.
First of all, the space of smooth differential forms with the inner product $\langle,\rangle$ is not a Hilbert space, i.e. it is not complete. This is just as for C^∞ functions on an interval - they do not form a complete linear topological space. In both cases, the completed space (equivalence classes of Cauchy sequences) consists of L^2-integrable objects. But then, we cannot differentiate them.

Put in another way: after completing $\Omega^k M$, we can find element of smallest norm in the *closure of every cohomology class*. But why should such an element be a smooth differentiable form?

The solution is to complete $\Omega^k M$ not with respect to the norm $||\phi|| = \langle\phi, \phi\rangle^{1/2}$, but one that involves also integration of (covariant) derivatives up to high order s. Such a completion is called a Sobolev space. Let us denote it by $W^k_s M$. It is a Hilbert space, and the Laplacian extends to a *Fredholm* operator $\Delta_s : W^k_s(M) \to W^k_{s-2}(M)$ (Fredholm = $\dim \operatorname{Ker} \Delta < +\infty$, $\dim \operatorname{Coker} \Delta < +\infty$). Moreover $\operatorname{Ker} \Delta_s = \operatorname{Ker} \Delta$, so that every "Sobolev class" harmonic form is actually smooth.
We have now a well-defined $Y = (\operatorname{Ker} \Delta_s)^\perp$ and what we need is to show that $Y \cap \Omega^k M = d\Omega^{k-1}M \oplus d^*\Omega^{k+1}M$. We have $Y = \Im \Delta^*_s : W^k_{s-2}(M) \to W^k_s(M)$, but on smooth forms, $\Delta^*_s = \Delta$ (Δ is self-adjoint) so any smooth form ψ orthogonal to $\operatorname{Ker} \Delta$ is in the image of Δ, i.e. $\psi = \Delta u = (dd^* + d^*d)u = d(d^*u) + d^*(du)$.
I hope that this rough outline gives some idea why the Hodge-de Rham theorem is true!

We now wish to an analogous decomposition on a compact Hermitian manifold (M,h,J) using the operator $\bar\partial$. Thus, we define the formal adjoint $\bar\partial^* : \Omega^{p,q+1}M \longrightarrow \Omega^{p,q}M$ of $\bar\partial$ and the complex Laplacian $\Delta_{\bar\partial} = \bar\partial^*\bar\partial + \bar\partial\bar\partial^*$. The main points are:

- We have a hermitian inner product on $\Omega^{p,q}M$.
- There is a natural orientation on a complex manifold.
- The Hodge star maps $\Omega^{p,q}M$ to $\Omega^{n-q,n-p}M$.
- As n is even, $*^2 = (-1)^{p+q}$.
- $\bar\partial^* = -*\partial*$.

The formula for $\Delta_{\bar\partial}$ in local coordinates is similar to the one given for the Riemannian Laplacian: if $h = \mathrm{Re}\sum h_{ij} dz_i \otimes dz_j$, then (on functions)

$$\Delta_{\bar\partial} f = -\frac{1}{\sqrt{|h|}} \sum_{i,j} \frac{\partial}{\partial z_i}\left(\sqrt{|h|}\, h^{ij} \frac{\partial f}{\partial \bar z_j}\right).$$

But note the crucial difference!

A differential form ϕ, such that $\Delta_{\bar\partial}\phi = 0$ is called $\bar\partial$-harmonic. Just like for Δ_g: if M is compact, then ϕ is $\bar\partial$-harmonic iff $\bar\partial\phi = 0$ and $\bar\partial^*\phi = 0$.

We denote by $\mathcal{H}^{p,q}M$ the space of $\bar\partial$-harmonic forms of type (p,q).

Theorem (Dolbeaut decomposition theorem)

On a compact Hermitian manifold (M,h,J):

$$\Omega^{p,q}M = \mathcal{H}^{p,q}(M) \oplus \bar\partial\Omega^{p,q-1}M \oplus \bar\partial^*\Omega^{p,q+1}M,$$

where the summands are orthogonal with respect to the global hermitian product $\langle , \rangle$.

The proof is similar to that of the Hodge-de Rham theorem.; Again:

Corollary (Dolbeaut isomorphism)

The natural map $\mathcal{H}^{p,q}(M) \longrightarrow H^{p,q}(M)$, given by $\psi \mapsto [\psi]$ is an isomorphism.

Corollary (Serre duality)

On a compact complex n-dimensional manifold M, $H^{p,q}(M) \simeq H^{n-p,n-q}(M)$.

Proof. $\bar{*}\mathcal{H}^{p,q}(M) \longrightarrow \mathcal{H}^{n-p,n-q}(M)$ is an isomorphism. ☐

Thus, on a Hermitian manifold (M, g, I) we have two Laplacians: the Riemannian Δ_h and the complex one $\Delta_{\bar{\partial}}$. In general, there is no relation between them; hence no relation between harmonic and $\bar{\partial}$-haarmonic forms; hence no relation between De Rham and Dolbeault cohomology.
A miracle of Kähler geometry:

Proposition

If (M, h, J) is a Kähler manifold, then $\Delta_h = 2\Delta_{\bar{\partial}}$.

Proof. Both Laplacians, written in local coordinates, involve only first derivatives of the metric. Thus, in normal Kähler coordinates (complex coordinates in which the metric is *Euclidean* + $O(|z|^2)$ at a point p, the two Laplacians are $\Delta_{\mathrm{EUCL}} + O(|z|)$ and $\Delta_{\bar{\partial}-\mathrm{EUCL}} + O(|z|)$. Since the Proposition holds in $\mathbb{C}^n$, we have $\Delta_{h|p} = 2\Delta_{\bar{\partial}|p}$ and, hence, everywhere. □

On a *compact* Kähler manifold we obtain now Hodge relations from this Proposition, the Hodge-De Rham and the Dolbeault decomposition theorems.