ADMISSIBLE PROOF THEORY AND BEYOND

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1 Prologue

Ordinals made their entrance in proof theory through Gentzen’s second consistency proof for Peano Arithmetic by transfinite induction up to ε_0 , the latter being applied only to decidable predicates (cf. Gentzen [1938]). Gentzen’s constructive use of ordinals as a method of analyzing formal theories has come to be a paradigm for much of proof theory from then on, particularly as exemplified in the work of Schütte, Takeuti and their schools.¹

One of the strongest theories for which ordinal-theoretic bounds have been obtained is the impredicative subsystem of second order arithmetic based on Δ_2^1 comprehension plus bar induction. The latter result was achieved by employing the most advanced techniques in this area of research: cut elimination for infinitary calculi of ramified set theory with Π_2 -reflection rules. This gathering of tools was entitled “Admissible Proof Theory” (cf. Pohlers [1982]), yet another appropriate title could have been “Proof Theory of Π_2 -Reflection”. Unfortunately, these methods are not strong enough for carrying through an ordinal analysis of Π_2^1 comprehension, let alone for second order arithmetic.

This article will survey the state of the art nowadays, in particular recent advance in proof theory beyond admissible proof theory, giving some prospects of success of obtaining an ordinal analysis of Π_2^1 comprehension.

Although a great deal of ordinally informative proof theory has been pursuing an extension of Hilbert’s program, that is sought-for consistency proofs, I shall only indulge very little in this issue.² Even those who wish to detach themselves from consistency matters may benefit from ordinal analyses. Ordinal analysis has proved to be an important tool in reductive proof theory and also for the determination of the provably total functions of various complexities of a variety of theories. Putting things into a broader perspective, a leit-motif for ordinal analysis could have been Kreisel’s question:

What more than its truth have we recognized, when we have established a theorem in a formal theory?

The article is divided into four parts. In Section 2 I roughly describe the role of ordinal and ordinal analysis in proof theory.

¹cf. Schütte [1977], Takeuti [1987], Pohlers [1987], Pohlers [1991].

²For details cf. Takeuti [1987] and also the papers Feferman [1988] and Sieg [1988] being written on the occasion of a special Symposium on Hilbert’s Program.

Section 3 will be concerned with the program of admissible proof theory as well as its achievements. Also, in a nutshell, the cut-elimination procedure for Kripke-Platek set theory is given.

After having witnessed a real ordinal analysis, the reader will be more prepared for a discussion of the many facets of ordinal analysis which will be the purpose of Section 4.

The final Section 5 deals with new cut-elimination procedures for reflections higher than Π_2 . Cut-elimination for Π_n -reflection entails a proof-theoretic treatment of theories of nonmonotone inductive definitions. It is also touched upon the question of how far afield all this is from Π_2^1 comprehension.

2 Ordinal analysis

Let T be a theory the language of which is rich enough to contain formulas expressing well-foundedness properties. In addition, assume that T comprises primitive recursive arithmetic PRA and that T is faithful, i.e. whenever $T \vdash A$, then A is true. Under these conditions the *proof-theoretic ordinal* $|T|$ of T is often defined as follows:

$$|T| = \sup \{ \alpha : \alpha \text{ provably recursive in } T \},$$

where α is said to be provably recursive in T if there is a recursive well-ordering $\prec$ with order-type α such that

$$T \vdash WO(\prec)$$

with $WO(\prec)$ expressing in the language of T that $\prec$ is a well-ordering.

The determination of $|T|$ is then called *ordinal analysis* of T .

The above definition of $|T|$ has the advantage of being mathematically precise, but as to the activity named 'ordinal analysis' it is left completely open what constitutes such an analysis and in what terms $|T|$ is to be given.

A nude set-theoretical ordinal is hardly ever of interest from the viewpoint of foundation. In proof theory attention focusses on structured ordinals which can be dealt with in a finitary manner. The paradigm is Gentzen's use of Cantor's representation of ordinals $< \varepsilon_0$. Every ordinal $0 < \alpha < \varepsilon_0$ has a unique representation of the form

$$\alpha = \omega^{\alpha_1} + \dots + \omega^{\alpha_n}$$

with $\alpha_n \leq \dots \leq \alpha_1 < \alpha$.

Therefore the ordinals $< \varepsilon_0$ can be represented by terms built up from a symbol for 0 and symbols for the function $+$ and $\lambda \xi. \omega^\xi$. We also gain finitary control on such (infinite) ordinals because of the following facts:

- For every expression E composed of the symbols $0, +, \omega$ it can be decided whether E represents an ordinal.
- Given two representations E_0, E_1 of ordinals α_0, α_1 respectively, we know how to compare α_0 and α_1 solely by means of the build-up of E_0 and E_1 .

It is by now clear that $|T|$ is to be given in terms of a system of ordinal representation usually called ordinal notation system. Significant features that ordinal notation systems should have will be addressed in Section 4.

I have always found the description of ordinal analysis as a quest for proof-theoretic ordinals to be bad propaganda, above all, since it remains silent about the most interesting aspects of ordinal analysis and prejudices people against this enterprise. If experience has shown that the ordinal $|T|$ is intrinsically related to the proof power of T , it is rarely the sheer knowing of $|T|$ that lends itself to important information about T .³ Most of the vital information springs from the proof itself. Turning attention to practice, an ordinal analysis of T provides, among others, the following results:

- A reduction of T to Heyting's Arithmetic, HA , plus a scheme of transfinite induction.
- A consistency proof of T .
- A classification of the provably recursive functions (on $\mathbb{N}$) of T .
- A classification of the provably hyperarithmetical functions of T .
- A classification of the provably Δ_2^1 functions of T .
- A description of partial models of T , for instance models of all Π_2^1 or Π_3^1 theorems of T .

A discussion of these points will be more fruitful and lively after the reader has gained some experience with ordinal analysis, so we defer it to the next but one Section.

3 Admissible proof theory

Admissible proof theory arose out of the work of Jäger und Pohlers (cf. Pohlers [1982], Pohlers [1987]) who from a proof-theoretic stance started to investigate weak set theories featuring admissible sets. The direct proof-theoretical treatment of set theories is rather recent. Historically the primary concern has been on subsystems of second order arithmetic and theories of iterated inductive definitions (cf. Buchholz et al. [1981]).

Admissible sets are the transitive models of a remarkable subsystem of ZF , known as Kripke–Platek set theory (hereinafter called KP). Admissible sets were a major source of interaction between model theory, recursion theory and set theory (cf. Barwise [1975]).

In this section I am going to sketch an ordinal analysis for KP . The motivation behind this is twofold. On the one hand, I would like to give some insight into admissible proof theory by presenting the basic ideas that underly its cut-elimination procedures. On the other hand, this will serve as a foil for a comparison with new cut-elimination procedures in Section 5 and also for the discussion in Section 4.

³Actually, it has to be reckoned with theories where the proof-theoretic ordinal in the above sense doesn't reflect the proof-theoretic strength of the theory.

3.1 The system KP

Though considerably weaker than ZF , a great deal of set theory requires only the axioms of KP . The axioms of KP are:⁴

<i>Extensionality:</i>	$a = b \rightarrow [F(a) \leftrightarrow F(b)]$ for all formulas F .
<i>Foundation:</i>	$\exists x G(x) \rightarrow \exists x [G(x) \wedge (\forall y \in x) \neg G(y)]$
<i>Pair:</i>	$\exists x (x = \{a, b\})$.
<i>Union:</i>	$\exists x (x = \bigcup a)$.
<i>Infinity:</i>	$\exists x [x \neq \emptyset \wedge (\forall y \in x)(\exists z \in x)(y \in z)]$. ⁵
Δ_0 Separation:	$\exists x (x = \{y \in a : F(y)\})$ ⁶ for all Δ_0 -formulas F in which x does not occur free.
Δ_0 Collection:	$(\forall x \in a) \exists y G(x, y) \rightarrow \exists z (\forall x \in a) (\exists y \in z) G(x, y)$ for all Δ_0 -formulas G .

By a Δ_0 formula we mean a formula of set theory in which all the quantifiers appear restricted, that is have one of the forms $(\forall x \in b)$ or $(\exists x \in b)$.

KP arises from ZF by completely omitting the power set axiom and restricting separation and collection to absolute predicates (cf. Barwise [1975]), i.e. Δ_0 formulas. These alterations are suggested by the informal notion of ‘predicative’. KP is an impredicative theory, notwithstanding. It is known from Howard [1968],[1981] and Jäger [1982] that KP proves the same arithmetic sentences as Feferman’s system ID_1 of positive inductive definitions (cf. Feferman [1970]). Its proof-theoretic ordinal is the Howard–Bachmann ordinal $\theta_{\varepsilon_{\Omega+1}0}$.

3.2 Infinitary calculi

Peano Arithmetic, PA , does not admit cut-elimination. However, it is well known that the infinitary calculus PA_ω which results from PA by replacing the induction scheme by the so-called ω -rule

$$\frac{\Gamma, A(\bar{n}) \text{ for all } n}{\Gamma, \forall x A(x)}$$

does admit cut-elimination.⁷ An ordinal analysis for PA is then attained as follows:

- Each PA -proof can be unfolded into a PA_ω -proof of the same sequent.

⁴For technical convenience, $\in$ will be taken to be the only predicate symbol of the language of set theory. This does no harm, since equality can be defined by $a = b :\Leftrightarrow (\forall x \in a)(x \in b) \wedge (\forall x \in b)(x \in a)$, provided that we state extensionality in a slightly different form than usually.

⁵ $x = \{y \in a : F(y)\}$ stands for the Δ_0 -formula $(\forall y \in x)[y \in a \wedge F(y)] \wedge (\forall y \in a)[F(y) \rightarrow y \in x]$.

⁶This contrasts with Barwise [1975] where Infinity is not included in KP .

⁷ $\bar{n}$ stands for the n^{th} numeral.

- Each such PA_ω -proof can be transformed into a cut-free PA_ω -proof of the same sequent of length $< \varepsilon_0$.

In order to get a similar result for KP , we have to work a bit harder.

Experience has shown that the main obstacle for understanding ordinal analysis of impredicative theories is raised by its being intimately linked to specific systems of ordinal notations, even worse, to auxiliary deduction functions or relations needed in order for this method to work (cf. Pohlers [1981]).⁸ Fortunately, Buchholz [1991] has presented a new approach which is distinguished by conceptual clarity and flexibility, and in particular by the fact that its basic concepts are in no way related to any system of ordinal notations. We are going to take up Buchholz's approach but in an even more relaxed atmosphere, thereby refraining from technical details as far as possible. Especially, we shall put forward that the collapsing of proof trees which is paramount in impredicative proof theory can be understood in terms of the usual Mostowski collapse familiar from set theory.

At the outset, we set up an infinitary calculus of ramified set theory which is modelled upon the constructible hierarchy.

For α an ordinal, L_α is the α^{th} level of Gödel's constructible hierarchy, i.e.

$$L_0 = \emptyset$$

$$L_\alpha = \bigcup_{\beta < \alpha} L_\beta \quad \text{if } \alpha \text{ is a limit ordinal}$$

$$L_\alpha = \{X : X \subseteq L_\beta \text{ and } X \text{ is definable over } \langle L_\beta, \in \rangle\} \quad \text{if } \alpha = \beta + 1.$$

Guided by the analogy with PA_ω , we would like to invent an infinitary rule which when added to KP enables us to eliminate cuts. However, as opposed to the natural numbers, it is not very clear how to bestow upon each element of the set-theoretic universe a name that reflects its generation; but within the confines of the constructible universe which is made from the ordinals it is pretty obvious how to name sets once we have given names to ordinals. Thus we are naturally led to the calculus RS we are going to introduce next.

3.2.1 Infinitary syntax

RS -terms and their levels are inductively defined as follows.

1. For every ordinal α , $\check{L}_\alpha$ is an RS -term of level α .
2. If $F(x, y_1, \dots, y_n)$ is a formula of set theory with no free variables other than shown, and $s_1, \dots, s_n$ are RS -terms of levels $< \alpha$, then the formal expression $[x \in \check{L}_\alpha : F(x, s_1, \dots, s_n)]^{\check{L}_\alpha}$ is an RS -term of level α .

⁸For instance, several years ago in a seminar at Münster devoted to the ordinal analysis of ν -fold iterated inductive definitions, more than half of the time was spent on developing collapsing functions with peculiar features, and those collapsing functions were not the ones that surfaced in the corresponding notation system.

We denote the level of an RS -term t by $|t|$. For a formula F , we denote by F^a the formula that is obtained by restricting any unbounded quantifier in F by a .

The interpretation i of an RS -term in L is, as was to expected,

- $i(\check{L}_\alpha) = L_\alpha$
- $i([x \in \check{L}_\alpha : F(x, s_1, \dots, s_n)^{\check{L}_\alpha}]) = \{x \in L_\alpha : L_\alpha \models F(x, i(s_1), \dots, i(s_n))\}$
 $= \{x \in L_\alpha : L \models F(x, i(s_1), \dots, i(s_n))^{L_\alpha}\}.$

An RS -formula is one that arises from a Δ_0 formula of set theory by replacing all its free variables with RS -terms. Let G be an RS -formula. By way of the interpretation i , validity of G in L , $L \models G$, is understood.

Abbreviations.

$$k(G) = \{\alpha : \check{L}_\alpha \text{ occurs in } G\} \quad (\text{subterms included}).$$

$$|G| = \sup k(G).$$

For RS -terms a, b with $|a| < |b|$, $\diamond$ a propositional junctor, and A an arbitrary RS -formula, we set

$$(a \check{e} b) \diamond A = \begin{cases} B(a) \diamond A & \text{if } b \equiv [x \in \check{L}_\beta : B(x)] \\ A & \text{if } b \equiv \check{L}_\beta. \end{cases}$$

Obviously $(a \in b) \diamond A$ and $(a \check{e} b) \diamond A$ have the same truth-value.

3.2.2 Infinitary rules

Next we introduce an infinitary sequent calculus, RS , that admits cut elimination. $A, B, C, \dots, F(t), G(t), \dots$ range over RS -formulas. We denote by upper case Greek letters $\Gamma, \Delta, \Lambda, \dots$ finite sets of RS -formulas. The intended meaning of $\Gamma = \{A_1, \dots, A_n\}$ is the disjunction $A_1 \vee \dots \vee A_n$. Γ, A stands for $\Gamma \cup \{A\}$.

The rules of RS are:

- ($\wedge$) $\frac{\Gamma, A \quad \Gamma, A'}{\Gamma, A \wedge A'}$
- ($\vee$) $\frac{\Gamma, A_i}{\Gamma, A_0 \vee A_1} \text{ if } i \in \{0, 1\}$
- ($\forall$) $\frac{\Gamma, s \check{e} t \rightarrow F(s) \text{ for all } s \text{ such that } |s| < |t|}{\Gamma, (\forall x \in t) F(x)}$
- ($\exists$) $\frac{\Gamma, s \check{e} t \wedge F(s)}{\Gamma, (\exists x \in t) F(x)} \text{ if } |s| < |t|$

$$\begin{aligned}
(\not\in) \quad & \frac{\Gamma, s \check{\in} t \rightarrow r \neq s \text{ for all } s \text{ such that } |s| < |t|}{\Gamma, r \not\in t} \\
& \text{(where } r \neq s := \neg(r = s) \text{ and } r \not\in t := \neg(r \in t)) \\
(\in) \quad & \frac{\Gamma, s \check{\in} t \wedge r = s}{\Gamma, r \in t} \text{ if } |s| < |t| \\
(Cut) \quad & \frac{\Gamma, A \quad \Gamma, \neg A}{\Gamma}.
\end{aligned}$$

As in Schwichtenberg [1977] we shall regard $\neg$ in front of a non-atomic formula as a defined operation:

$\neg A$ is defined to be the formula obtained from A by (i) putting a $\neg$ in front of any atomic formula, (ii) replacing $\wedge, \vee, (\forall x \in a), (\exists x \in a)$ by $\vee, \wedge, (\exists x \in a), (\forall x \in a)$, respectively, and (iii) dropping double negations.

Owing to the symmetry of the pairs of rules

$$\begin{aligned}
& (\wedge), (\vee) \\
& (\forall), (\exists) \\
& (\not\in), (\in),
\end{aligned}$$

the usual cut-elimination procedure (cf. Schwichtenberg [1977]) applies to RS . But unequal to the situation for PA and PA_ω , RS does not allow of any nontrivial embedding of KP ; the trivial one being provided by the fact that for any admissible ordinal κ , L_κ is a model of KP and the following completeness property of RS :

Theorem. (cf. Pohlers [1991], Theorem 3.2.6) *For each RS -formula G , if $L \models G$, then there is a RS proof of G .*

The only axioms of KP that shatter hopes of obtaining an informative embedding into RS are instances of Δ_0 collection. To remedy this, we simply add a new rule to RS which plainly entails Δ_0 collection. The reverse of the medal is that we need to be particular about permitting derivations in order to restore (partial) cut-elimination.

In the sequel, we fix an admissible ordinal Ω . Henceforth we will only be concerned with RS_Ω -formulas, i.e. RS -formulas of the form $F(s_1, \dots, s_n)^{\check{L}_\Omega}$, where $s_1, \dots, s_n$ are RS -terms of levels $< \Omega$ and $F(x_1, \dots, x_n)$ is a formula of set theory. In case that $F(x_1, \dots, x_n)$ contains no unbounded universal quantifiers, $F(x_1, \dots, x_n)$ is said to be a Σ formula, and $F(s_1, \dots, s_n)^{\check{L}_\Omega}$ will be called $\Sigma(\Omega)$ formula. Frequently we write A^α instead of $A^{\check{L}_\alpha}$. Occasionally, $(\exists x^\alpha)$ will be a shorthand for $(\exists x \in \check{L}_\alpha)$.

The already announced rule is

$$(\Sigma\text{-Ref}_\Omega) \quad \frac{\Gamma, A^\Omega}{\Gamma, (\exists z \in \check{L}_\Omega) A^z} \text{ if } A^\Omega \text{ is } \Sigma(\Omega).$$

The motivation behind this rule is that on the basis of the other axioms of KP , Δ_0 collection is equivalent to the scheme of Σ reflection, i.e.

$$B \rightarrow \exists z B^z$$

for every Σ formula B (cf. Barwise [1975]).

3.2.3 $\mathcal{H}$ -controlled derivations

The concept of $\mathcal{H}$ -controlled derivations stems from Buchholz [1991].

Let $P(ON) = \{X : X \text{ is a set of ordinals}\}$.

A class function

$$\mathcal{H} : P(ON) \rightarrow P(ON)$$

will be called operator if the following conditions are satisfied for $X, X' \in P(ON)$:

(H1) $0 \in \mathcal{H}(X)$. For $\alpha = \omega^{\alpha_1} + \dots + \omega^{\alpha_n}$ with $\alpha_1 \geq \dots \geq \alpha_n$, it holds $\alpha \in \mathcal{H}(X)$ if and only if $\alpha_1, \dots, \alpha_n \in \mathcal{H}(X)$. (Especially, $\mathcal{H}(X)$ is closed with respect to $+$ and $\lambda\xi.\omega^\xi$, i.e., if $\alpha, \beta \in \mathcal{H}(X)$, then $\alpha + \beta, \omega^\alpha \in \mathcal{H}(X)$.)

(H2) $X \subseteq \mathcal{H}(X)$

(H3) $X' \subseteq \mathcal{H}(X) \Rightarrow \mathcal{H}(X') \subseteq \mathcal{H}(X)$.

Abbreviations. $\alpha \in \mathcal{H} := \alpha \in \mathcal{H}(\emptyset)$

$X \subseteq \mathcal{H} := X \subseteq \mathcal{H}(\emptyset)$

For an RS_Ω -term s , $\mathcal{H}[s]$ denotes the operator $(X \mapsto \mathcal{H}(k(s) \cup X))_{X \in P(ON)}$.

Let $\Omega \in \mathcal{H}$ and Γ be a finite set of RS_Ω -formulas. The relation $\mathcal{H} \vdash^\alpha \Gamma$ ($\mathcal{H}$ -controlled derivability) is defined inductively by

$$\{\alpha\} \cup k(\Gamma) \subseteq \mathcal{H}$$

and the following rules

$$\begin{array}{ll}
(\wedge) & \frac{\mathcal{H} \vdash^{\alpha_0} \Lambda, A_0 \quad \mathcal{H} \vdash^{\alpha_1} \Lambda, A_1}{\mathcal{H} \vdash^\alpha \Lambda, A_0 \wedge A_1} \quad \alpha_0, \alpha_1 < \alpha \\
(\vee) & \frac{\mathcal{H} \vdash^{\alpha_0} \Lambda, C}{\mathcal{H} \vdash^\alpha \Lambda, A \vee B} \text{ if } C \in \{A, B\} \quad \alpha_0 < \alpha \\
(\forall) & \frac{\dots \mathcal{H}[s] \vdash^{\alpha_s} \Lambda, s \check{\in} t \rightarrow F(s) \dots (|s| < |t|)}{\mathcal{H} \vdash^\alpha \Lambda, (\forall x \in t) F(x)} \quad \alpha_s < \alpha \\
(\exists) & \frac{\mathcal{H} \vdash^{\alpha_0} \Lambda, s \check{\in} t \wedge F(s)}{\mathcal{H} \vdash^\alpha \Lambda, (\exists x \in t) F(x)} \text{ if } |s| < |t| \quad \alpha_0, |s| < \alpha, k(s) \subseteq \mathcal{H}
\end{array}$$

$$\begin{array}{ll}
(\not\in) & \frac{\cdots \mathcal{H}[s] \mid^{\alpha_s} \Lambda, s \check{\in} t \rightarrow r \neq s \cdots (|s| < |t|)}{\mathcal{H} \mid^{\alpha} \Lambda, r \not\in t} \quad \alpha_s < \alpha \\
(\in) & \frac{\mathcal{H} \mid^{\alpha_0} \Lambda, s \check{\in} t \wedge r = s}{\mathcal{H} \mid^{\alpha} \Lambda, r \in t} \text{ if } |s| < |t| \quad \alpha_0, |s| < \alpha, k(s) \subseteq \mathcal{H} \\
(\Sigma\text{-Ref}_{\Omega}) & \frac{\mathcal{H} \mid^{\alpha_0} \Lambda, A^{\Omega}}{\mathcal{H} \mid^{\alpha} \Lambda, (\exists z \in \check{L}_{\Omega}) A^z} \text{ if } A^{\Omega} \in \Sigma(\Omega), \quad \alpha_0 < \alpha \\
(Cut) & \frac{\mathcal{H} \mid^{\alpha_0} \Lambda, B \quad \mathcal{H} \mid^{\alpha_0} \Lambda, \neg B}{\mathcal{H} \mid^{\alpha} \Lambda} \quad \alpha_0 < \alpha.
\end{array}$$

Since we also want to keep control of the cuts of $\mathcal{H}$ -controlled derivations, we assign a rank, $rk(A)$, to RS_{Ω} -formulas A . All we need to know is that $rk(A) = \omega \cdot |A| + n$ for some $n < \omega$, $rk(A) = rk(\neg A)$, and $rk((\exists z \in \check{L}_{\Omega}) A^z) = \Omega$ if $A \in \Sigma(\Omega)$.

We write $\mathcal{H} \mid_{\rho}^{\alpha} \Gamma$ to express that there is an $\mathcal{H}$ -controlled derivation of Γ such that $rk(B) < \rho$ holds for all cut formulas B in this derivation.

Having defined $\mathcal{H}$ -controlled derivability, the notion of an $\mathcal{H}$ -controlled derivation (or proof) is understood. To be more precise, an $\mathcal{H}$ -controlled derivation is a well-founded tree the nodes of which are pairs $\langle \alpha, \Gamma \rangle$ resulting from its immediate successor nodes by one of the above rules.

We will use the notation $\Pi_{\mathcal{H}} \mid_{\varrho}^{\alpha} \Gamma$ to indicate that $\Pi_{\mathcal{H}}$ is an $\mathcal{H}$ -controlled derivation witnessing $\mathcal{H} \mid_{\varrho}^{\alpha} \Gamma$.

3.3 Embedding KP

Let d be a KP -proof of a sentence F . Then there exists an integer n such that for every operator $\mathcal{H}$ with $\Omega \in \mathcal{H}$ we have an $\mathcal{H}$ -controlled derivation

$$\Pi_{\mathcal{H}, \Omega} \mid_{\Omega+n}^{\Omega \cdot n} F^{\Omega}$$

(cf. Buchholz [1991]). Furthermore, it is to be noted that the construction of the $\mathcal{H}$ -proof $\Pi_{\mathcal{H}, \Omega}$ of F^{Ω} is uniform in $\mathcal{H}$ and Ω . This is reflected by the following facts: If $\mathcal{H}'$ majorizes $\mathcal{H}$, i.e. $\forall X (\mathcal{H}(X) \subseteq \mathcal{H}'(X))$, then $\Pi_{\mathcal{H}, \Omega} = \Pi_{\mathcal{H}', \Omega}$. If $\Omega < \hat{\Omega} \in \mathcal{H}$, then $\Pi_{\mathcal{H}, \Omega}$ and $\Pi_{\mathcal{H}, \hat{\Omega}}$ are closely related to each other. $\Pi_{\mathcal{H}, \Omega}$ can be obtained from $\Pi_{\mathcal{H}, \hat{\Omega}}$ by the following pruning and substitution processes:

- Omit from each instance of a rule

$$\frac{\cdots \Delta, H(t) \cdots (|t| < \hat{\Omega})}{\Delta, H(x)}$$

in $\Pi_{\mathcal{H}, \hat{\Omega}}$ all premisses $\Delta, H(t)$ with $\hat{\Omega} \leq |t|$ as well as the subproofs of these premisses.

- Within the multilated proof, each ordinal $\alpha > 0$ has Cantor normal form $\alpha = \hat{\Omega}^{k_1}\beta_1 + \dots + \hat{\Omega}^{k_r}\beta_r$ where $k_1 > \dots > k_r$ and $\beta_1, \dots, \beta_r < \Omega$. Now replace α by $\Omega^{k_1}\beta_1 + \dots + \Omega^{k_r}\beta_r$.

The use of a whole family of proofs is reminiscent of Girard's notion of β -proof (cf. Girard [1985]).

Indeed, there are more points of contact. Usually for a single $\mathcal{H}$, it will not be possible to transform an $\mathcal{H}$ -proof into a cut-free $\mathcal{H}$ -proof. To overcome this difficulty, we pass over to stronger and yet stronger operators during the cut-elimination procedure, but in a controlled manner, thereby working simultaneously on a whole family of proofs indexed by operators.

3.4 Cut-elimination

As already mentioned, $(\Sigma\text{-Ref}_\Omega)$ is the only rule that spoils cut-elimination. Since an instance of $(\Sigma\text{-Ref}_\Omega)$ always introduces a formula of rank Ω , we can at least remove all cuts of rank $> \Omega$. So we get

Cut-elimination I. *Let $n > 0$. Then:*

$$\mathcal{H} \Big|_{\Omega+n}^{\Omega \cdot n} \Gamma \Rightarrow \mathcal{H} \Big|_{\Omega+1}^{\Omega(n)} \Gamma,$$

where $\Omega(1) := \Omega$ and $\Omega(k+1) := \Omega^{\Omega(k)}$.⁹

A first step towards elimination of $(\Sigma\text{-Ref}_\Omega)$ is provided by the following

Bounding Theorem. *Let B^Ω be a $\Sigma(\Omega)$ formula. If $\alpha < \Omega$ and $\mathcal{H} \Big|_\rho^\alpha \Gamma, B^\Omega$, then $\mathcal{H} \Big|_\rho^\alpha \Gamma, B^\alpha$.*

This result is easily proved by induction on α . First let us focus on the case when the last inference is $(\Sigma\text{-Ref}_\Omega)$ with principal formula B^Ω . Then B^Ω is of the form $(\exists z \in \check{L}_\Omega)A^z$, and we have $\mathcal{H} \Big|_\Omega^{\alpha_0} \Gamma, A^{\alpha_0}$ for some $\alpha_0 < \alpha$. By induction hypothesis we get

$$\mathcal{H} \Big|_\Omega^{\alpha_0} \Gamma, A^{\alpha_0},$$

which is the same as $\mathcal{H} \Big|_\Omega^{\alpha_0} \Gamma, \check{L}_{\alpha_0} \check{\in} \check{L}_\Omega \wedge A^{\check{L}_{\alpha_0}}$, thus $\mathcal{H} \Big|_\Omega^\alpha \Gamma, (\exists z \in \check{L}_\Omega)A^z$ follows by an inference $(\exists)$.

The key to an understanding of the Boundedness Theorem is provided by the case when the last inference is of the form

$$\frac{\mathcal{H} \Big|_\Omega^{\alpha_0} \Gamma, F(s)^\Omega, A}{\mathcal{H} \Big|_\Omega^\alpha \Gamma, (\exists x \in \check{L}_\Omega)F(s)^\Omega} \quad (\exists)$$

⁹This is the right place to explain why we demanded $\mathcal{H}(X)$ to be closed under $+$ and $\alpha \mapsto \omega^\alpha$: simply because these closure properties are needed for the above cut-elimination method.

with $A \equiv (\exists x \in \check{L}_\Omega) F(x)$. Using the induction hypothesis we then get

$$\mathcal{H} \left| \frac{\alpha_0}{\Omega} \right. \Gamma, F(s)^{\check{L}_\alpha}, A^\alpha.$$

The conditions imposed by $(\exists)$, ensure that $|s| < \alpha$, thus $\mathcal{H} \left| \frac{\alpha_0}{\Omega} \right. \Gamma, A^\alpha$ via an inference $(\exists)$.

The Boundedness Theorem also traces out the way for an elimination of $(\Sigma\text{-Ref}_\Omega)$ in the more general situation when $\mathcal{H} \left| \frac{\beta}{\Omega} \right. \Gamma$ with $\beta \geq \Omega$. However, we can no longer deal with arbitrary operators. In the sequel we shall restrict ourselves to operators $\mathcal{H}$ such that for each $\mathcal{H}$ -controlled derivation $\Pi_{\mathcal{H}}$ without Ω -branchings the following "collapsing" properties are satisfied:

(C1) *The set $\{ |s| < \Omega : s \text{ occurs in } \Pi_{\mathcal{H}} \}$ is bounded below Ω*

(C2) *The Mostowski collapse of the set $\{ \alpha : \alpha \text{ occurs in } \Pi_{\mathcal{H}} \}$ is less than Ω .*

Of course, requiring that $\Pi_{\mathcal{H}}$ has no Ω -branchings is a necessary condition for (C1) and (C2) to hold. But the reader might have a suspicion that the restrictions imposed by (C1) and (C2) give a too narrow class of operators in order for the cut-elimination to work. At the end of this Section we shall deliver a class of operators that fulfills (C1), (C2) and, in addition, is sufficiently rich for the purpose of cut-elimination.

Now let us fix an $\mathcal{H}$ -controlled derivation $\Pi_{\mathcal{H}} \left| \frac{\alpha}{\Omega} \right. \Gamma$ without Ω -branchings. This is for instance guaranteed if Γ is a set of $\Sigma(\Omega)$ formulas. On the other hand, if Γ entails a formula D which contains a quantifier $(\forall x \in \check{L}_\Omega)$, then it can be shown that $\mathcal{H} \left| \frac{\alpha}{\Omega} \right. \Gamma \setminus \{D\}$, i.e. D can be dropped from the derivation. Thus the exclusion of Ω -branchings is almost equivalent to Γ being a set of $\Sigma(\Omega)$ formulas.

Henceforth we assume that Γ is a set of $\Sigma(\Omega)$ formulas.

We are going to transform $\Pi_{\mathcal{H}}$ into a proof-tree without instances of $(\Sigma\text{-Ref}_\Omega)$. To this end, using (C1) pick $\beta < \Omega$ such that

$$\{ |s| < \Omega : s \text{ occurs in } \Pi_{\mathcal{H}} \} \subseteq \beta.$$

By (C2), we then find an order preserving function

$$f : \beta \cup \{ \alpha : \alpha \text{ occurs in } \Pi_{\mathcal{H}} \} \longrightarrow \gamma$$

onto some $\gamma < \Omega$.

Next let $\Pi_{\mathcal{H}}^f$ denote the tree that results from $\Pi_{\mathcal{H}}$ by replacing every node $\langle \xi, \Gamma \rangle$ in $\Pi_{\mathcal{H}}$ by $\langle f(\xi), \Gamma \rangle$. If we now define $\mathcal{H}_\beta$ via the equation

$$\mathcal{H}_\beta(X) = \mathcal{H}(X \cup (\beta + 1)),$$

we may expect that $\Pi_{\mathcal{H}}^f \left| \frac{f(\alpha)}{\Omega} \right. \Gamma$ is an $\mathcal{H}_\beta$ -controlled derivation. Indeed, this is readily verified. Since $f(\alpha) < \Omega$, we can employ the technique of the Bounding Theorem to get rid of all instances of $(\Sigma\text{-Ref}_\Omega)$ in $\Pi_{\mathcal{H}}^f$. We just have to replace the transitions in $\Pi_{\mathcal{H}}^f$ that are under the command of $(\Sigma\text{-Ref}_\Omega)$ by suitable instances of $(\exists)$. So we come up with a derivation $\Pi_{\mathcal{H}[\beta]} \left| \frac{f(\alpha)}{\Omega} \right. \Gamma$ that no longer contains $(\Sigma\text{-Ref}_\Omega)$.

After having devised ways and means to remove $(\Sigma\text{-}Ref_\Omega)$ from derivations

$$\tilde{\Pi}_{\mathcal{H}} \mid_{\Omega}^{\alpha} \Gamma$$

with $\Gamma \subseteq \Sigma(\Omega)$, we may now attack the problem of removing cuts of rank Ω from derivations $\Pi_{\mathcal{H}} \mid_{\Omega+1}^{\alpha} \Gamma$.

The reason why the usual cut-elimination method fails for cuts with rank Ω is that it is too limited to treat a cut in the following context:

$$\frac{\frac{\Pi_{\mathcal{H}}^0 \mid_{\Omega}^{\xi_0} \Gamma, A^{\Omega}}{\Pi_{\mathcal{H}}^1 \mid_{\Omega}^{\xi} \Gamma, (\exists z \in \check{L}_{\Omega}) A^z} (\Sigma\text{-}Ref_{\Omega}) \quad \frac{\cdots \Pi_{\mathcal{H}[s]} \mid_{\Omega}^{\xi_s} \Gamma, \neg A^s \cdots (|s| < \Omega)}{\Pi_{\mathcal{H}}^2 \mid_{\Omega}^{\xi} \Gamma, (\forall z \in \check{L}_{\Omega}) \neg A^z} (\forall)}{\Pi_{\mathcal{H}} \mid_{\Omega+1}^{\alpha} \Gamma} (Cut)$$

In this situation we are apt to apply the above introduced collapsing technique to $\Pi_{\mathcal{H}}^0$. Thus from $\Pi_{\mathcal{H}}^0$ we can extract a $\beta < \Omega$ and a function f such that $\mathcal{H}_{\beta} \mid_{\Omega}^{f(\xi_0)} \Gamma, A^{f(\xi_0)}$. Next we single out the $f(\xi_0)^{th}$ premiss of the last inference of $\Pi_{\mathcal{H}}^2$, that is

$$\Pi_{\mathcal{H}[f(\xi_0)]} \mid_{\Omega}^{\xi_{f(\xi_0)}} \Gamma, \neg A^{f(\xi_0)},$$

and, as $rk(A^{f(\xi_0)}) < \Omega$, a cut yields

$$\mathcal{H}_{\beta} \mid_{\Omega}^{\delta} \Gamma$$

for some δ .

In order to get rid of all cuts of rank Ω in an arbitrary derivation $\tilde{\Pi}_{\mathcal{H}} \mid_{\Omega+1}^{\alpha} \Gamma$, one has to repeat the foregoing process at worst a many times.

3.5 The functions θ_{α}

Yet another point is that we want to extract bounds from proofs of Σ formulas in KP . Therefore we have to take account of the quantitative aspects of “collapsing”. Specifically, the “seize” of the operator after reducing the cut-rank from $\Omega + 1$ to Ω has to be related (via a functional dependence) to the “seize” of the input operator.

Through the above construction of $\mathcal{H}_{\beta}$ from $\mathcal{H}$, one is quite naturally led to processes lying behind the construction of the Feferman–Aczel functions θ_{α} (cf. Schütte [1977]).

The functions $\theta_{\alpha} : \Omega \rightarrow \Omega$ are inductively generated as follows:¹⁰

Let

$$C(\alpha, \beta) = \begin{cases} \text{closure of } \{0, \Omega\} \cup \beta \\ \text{under } +, \xi \mapsto \omega^{\xi}, (\xi, \zeta \mapsto \theta_{\xi}(\zeta))_{\xi < \alpha, \zeta < \Omega} \end{cases}$$

¹⁰On the basis of the assumption $\Omega = \aleph_1$ (cf. Schütte [1977]) it is easily verified that $\theta_{\alpha}(\xi) < \Omega$ holds for $\xi < \Omega$ because of the countability of the set $C(\alpha, \xi)$. If, instead, Ω is merely supposed to be an admissible $> \omega$, it is by no means trivial to show that $\theta_{\alpha}(\xi) < \Omega$ (cf. Rathjen [1991b],[1991c]).

and

$$\theta_\alpha(\eta) = \eta^{\text{th}} \text{ ordinal } \delta \text{ such that } \delta \notin C(\alpha, \delta).$$

So this is a recursion with regard to α . If we now define operators $\mathcal{H}^\alpha$ by

$$\mathcal{H}^\alpha(X) = \bigcap \{C(\gamma, \beta) : X \subseteq C(\gamma, \beta) \wedge \alpha < \gamma\},$$

then the family of operators

$$(\mathcal{H}^\alpha)_{\alpha < \varepsilon_{\Omega+1}},$$

where $\varepsilon_{\Omega+1} = \sup_{n < \omega} \Omega(n)$, is sufficient for all our purposes. However, it takes some efforts to show that the operators $\mathcal{H}^\alpha$ ($\alpha < \varepsilon_{\Omega+1}$) meet the requirements (C1) and (C2). Moreover, the technical details of the cut-elimination procedure via the family $(\mathcal{H}^\alpha)_{\alpha < \varepsilon_{\Omega+1}}$ are very delicate and fiddly; but we shall be satisfied by having pointed out the key ideas.

3.6 Π_2 -reflection

As yet we have been dealing merely with Σ -reflection. One could argue that by doing so we covered Π_2 -reflection as well since Π_2 -reflection is a consequence of Σ -reflection, at least for structures of the form L_α (cf. Barwise [1975]). On the other hand if instead of $(\Sigma\text{-Ref}_\Omega)$ we incorporated the rule $(\Pi_2\text{-Ref}_\Omega)$ in the infinitary calculus, cut-elimination could be handled in almost the same spirit. By $(\Pi_2\text{-Ref}_\Omega)$ is meant the rule

$$\frac{\Gamma, \forall x^\Omega \exists y^\Omega F(x, y)}{\Gamma, \exists z [Tran(z) \wedge z \neq \emptyset \wedge (\forall x \in z)(\exists y \in z) F(x, y)]}$$

where $Tran(z)$ says that z is transitive and F ranges over the $\Delta_0(\Omega)$ -formulas.

At first glance it might be surprising that the collapsing technique of 3.4 also renders $(\Pi_2\text{-Ref}_\Omega)$ accessible since, as a rule, a derivation with instances of $(\Pi_2\text{-Ref}_\Omega)$ will have Ω -branchings whilst the collapsing technique is evidently constraint to derivation without such branchings. To overcome this difficulty, one employs an asymmetrical interpretation of the quantifiers. To explain this, let $\Pi_{\mathcal{H}} \mid_{\Omega} \Lambda$ be a derivation, possibly containing instances of $(\Pi_2\text{-Ref}_\Omega)$, and suppose that Λ is a set of RS_Ω -formulas of utmost complexity $\Pi_2(\Omega)$. Now proceed as follows:

- Pick $\gamma < \Omega$, and remove from each rule

$$\frac{\dots \Delta, H(t) \dots (|t| < \Omega)}{\Delta, \forall x^\Omega H(x)} (\forall)$$

in $\Pi_{\mathcal{H}}$ all the premisses $\Delta, H(t)$ with $\gamma \leq |t|$ as well as their subproofs. In the remaining tree replace every quantifier $\forall x^\Omega$ by $\forall x^\gamma$. For a suitably chosen operator (uniformly in γ) this will give a new proof without Ω -branchings to which thus the collapsing technique of 3.4 can be applied. After collapsing employ the Bounding Theorem to the collapsed derivation in order to extract a bound $g(\gamma) < \Omega$ for the existential quantifiers $(\exists x^\Omega)$.

- Compute the function g' which enumerates the fixed points of g .

- Construct a new operator $\mathcal{H}'$ from $\mathcal{H}$ which is closed under g' , i.e. $\eta \in \mathcal{H}'(X)$ entails $g'(\eta) \in \mathcal{H}'(X)$.
- By combining all the previous steps one receives an $\mathcal{H}'$ -controlled derivation $\Pi_{\mathcal{H}'} \mid_{\Omega} \Lambda$ without any instances of $(\Pi_2\text{-Ref}_{\Omega})$.

The final result reads as follows:

Theorem. *If $KP \vdash \forall x \exists y F(x, y)$ with F a Σ formula, then there exists an n such that for all $\xi < \Omega$,*

$$(\forall x \in L_{\xi})(\exists y \in L_{\theta_{\Omega(n)}(\xi+1)})F(x, y).$$

3.7 More admissibles

The cut-elimination procedure we have seen operating so well on KP can be adapted to extensions of the form

$$KP + \text{'there are many admissibles'}.$$

A prominent example for such a theory is Jäger's system KPi which, in addition to KP , has an inaccessibility axiom saying that for every set x there is an admissible set y containing it, i.e. $x \in y$.

It turned out that KPi is of the same proof-theoretic strength as the subsystem of second order arithmetic, $(\Delta_2^1 - CA) + BI$. The latter system consists of arithmetic plus

$$(\Delta_2^1 - CA) : \quad \forall n[F(n) \leftrightarrow G(n)] \rightarrow \exists X \forall n[n \in X \leftrightarrow F(n)]$$

$$\text{for all } F \in \Pi_2^1, \quad G \in \Sigma_2^1,$$

$$BI : \quad WO(<_X) \wedge \forall n[\forall m <_X n \, H(m) \rightarrow H(n)] \rightarrow \forall n H(n),$$

where $m <_X n := 3^m \cdot 5^n \in X$.

However, adjusting the methods which have been fruitfully employed to KP to KPi , is easier said than done. When ascending from KP to KPi , the ordinal notation systems as well as the cut-elimination procedures get more and more complicated. Notwithstanding that, the key idea pervades.

Finally, I shall briefly report on the theory KPM which is somewhat on the verge of admissible proof theory. KPM is designed to axiomatize essential features of a recursively Mahlo universe of sets, i.e. a universe that is a model of KPi and the scheme

$$(M) \quad \forall x \exists y H(x, y) \rightarrow \exists z[Ad(z) \wedge (\forall x \in z)(\exists y \in z)H(x, y)]$$

for all Δ_0 -formulas $H(a, b)$, where $Ad(z)$ signifies that z is an admissible set.

It is easily verified that L_{α} is a model of KPM if and only if α is a recursively Mahlo ordinal (cf. Hinman [1978]).¹¹

¹¹An admissible ordinal α is said to be recursively Mahlo if for every total function $f : \alpha \rightarrow \alpha$ that is Σ -definable in L_{α} there exists some $0 < \beta < \alpha$ such that $(\forall \xi < \beta)(f(\xi) < \beta)$.

An ordinal analysis for *KPM* was published in Rathjen [1991] and has also been obtained independently by Arai [1989].

Roughly speaking, the central scheme of *KPM* falls under the heading “ Π_2 -reflection with constraints”. The main stumbling block for an analysis of *KPM* was the invention of a suitable ordinal notation system. Till that time the recipes for creating ordinal notation systems had been based on ideas of Veblen and Bachmann. But these ideas only enabled one to engender collapsing functions which take as their values ordinals that, even when looked at from within the notation system, have cofinality ω , thus are highly singular ordinals. To be more precise, from the viewpoint of a notation System N the regularity of an ordinal $\kappa \in N$ is manifested by its being equipped with a collapsing function $\psi_\kappa : N \rightarrow N \cap \kappa$. Yet, in the approaches we have been just alluding to, the image of ψ_κ would never contain an ordinal π that is anew equipped with a collapsing function ψ_π , whereas the ordinal analysis of *KPM* requires a collapsing function always having this property. Eventually, such collapsing were developed in Rathjen [1990].

Not to leave any stone unturned, a characterization of *KPM* in terms of subsystems of second order arithmetic may be found in Rathjen [1991d]. It turns out that *KPM* proves the same sentences of second order arithmetic as $(\Delta_2^1 - CA) + BI$ augmented by an axiom schema expressing that every true Π_3^1 sentence (possibly including parameters) is already satisfied in a β -model of $(\Delta_2^1 - CA)$.

4 Aspects of ordinal analysis

This Section is reserved to the discussion of consequences of ordinal analysis which have been exhibited at the end of Section 2.

To explain these points, let $(D, \prec, \dots)$ be an ordinal notation system where D stands for a set of terms and $\prec$ denotes their ordering relation.¹² Moreover, let T be a theory which has been analyzed by way of $(D, \prec, \dots)$, resulting in $|T| = |\prec|$.

4.1 Consistency

By $PRWO(\prec)$ we mean the Π_2^0 -sentence of arithmetic expressing that $\prec$ is primitive recursively well-ordered, i.e. for every primitive recursive function p a strictly $\prec$ -descending chain $p(0) \prec p(1) \prec \dots$ must terminate after finitely many steps.

Then a consistency proof of T can be carried out in PRA extended by $PRWO(\prec)$.

PRA is distinguished here since it is widely agreed that this system does not go beyond finitary reasoning in Hilbert’s sense.

However, $PRA + PRWO(\prec)$ proves a much stronger consistency property, namely the 1-consistency of T , signifying that any Σ_1^0 sentence which is provable in T is also true.

As to PA , the result $PRA + PRWO(\varepsilon_0) \vdash Con(PA)$ can be easily drawn from Gentzen’s 1938 paper. There he assigned ordinal notations $ord(d) < \varepsilon_0$ to PA -derivations d and gave a primitive recursive reduction procedure R such that, for any derivation d of an inconsistency, $R(d)$ is also a derivation of an inconsistency and, in addition, $ord(R(d)) < ord(d)$.

¹²“...” is supposed to indicate that such a notation system usually conveys a much richer structure.

Later on, the ordinal ε_0 was reobtained as the ordinal of PA by use of derivations in infinitary logic with ω -rule, especially through Schütte's work. In the infinitary setting ordinals make a canonical appearance as a measure of the lengths of proof trees as well as of their cut-ranks. One is naturally led to ask whether Gentzen's result can also be achieved by employing cut-elimination for infinitary logic. This can be answered in the affirmative. It has turned out that primitive recursive proof-trees suffice and that the syntactical transformations employed in the course of cut-elimination can be represented by primitive recursive functions on the codes (cf. Schwichtenberg [1977]). Thus the use of infinitary derivations in the metamathematics is much in keeping with Gentzen's extension of the finite standpoint since the only principle for dealing with them that transcends finitistic means is a descending chain principle to show that certain 'concrete' (primitive recursive) processes terminate.

4.2 Reduction

$\prec$ will arise as the union of initial segments $\prec_n$ ($n \in \mathbb{N}$) such that, for any $n \in \mathbb{N}$, T proves $\prec_n$ being well-ordered.

Let $PA_{<|T|}$ stand for Peano Arithmetic endowed with the scheme of transfinite induction for all the orderings $\prec_n$. Then T is conservative over $PA_{<|T|}$ with respect to all arithmetic sentences or, equivalently, T is conservative over the intuitionistic system $HA_{<|T|}$ with respect to all arithmetic sentences modulo $\neg\neg$ translations.

Just to mention two applications of such reductions:

By an ordinal analysis of the theories ID_ν formalizing ν -fold iterated inductive definitions, Pohlers and Buchholz (cf. Pohlers [1981]) showed that these theories were reducible to their intuitionistic counterparts ID_ν^i .

Another famous example is provided by the reduction of Δ_2^1 comprehension plus bar induction to Feferman's constructive theory T_0 of functions and classes. T_0 is based on intuitionistic logic and is a suitable framework for Bishop style constructive mathematics. In 1977, Feferman (cf. Feferman and Sieg [1981]) had shown that T_0 is interpretable in $(\Delta_2^1 - CA) + BI$. The ordinal analysis of the latter system is due to joint work of Jäger and Pohlers [1982]. Jäger [1983] then showed that the well-ordering proof for any ordinal $< |(\Delta_2^1 - CA) + BI|$ can be carried out in T_0 ; thereby completing the reduction.

4.3 A classification of the provably recursive functions

The $\prec$ -descent recursive functions, $DCR(\prec)$, constitute the smallest class of recursive functions that has all the closure properties of the primitive recursive functions and, in addition, is closed with respect to the scheme:

If g and h are in the class, and there is some natural number k such that $h(x, y) \prec k$ holds for all $x, y \in \mathbb{N}$, then so is

$$f(m) = g(\mu n. [h(n, m) \preceq h(n+1, m)], m),$$

where μn indicates the least n in the ordering of the integers.

The reason for introducing the class $DCR(\prec)$ is (as was to be expected) that this class coincides with the provably recursive functions of T .

The concept of descent recursive function is for instance discussed in Smith [1985].

5 Beyond admissible proof theory

The strength of Π_2^1 comprehension is greatly bigger than that of Δ_2^1 comprehension. In particular, there is no way to describe this comprehension in terms of admissibility.

As to the set-theoretic side, Π_2^1 comprehension corresponds to Σ separation, i.e. the set of axioms

$$\exists z(z = \{x \in a : F(x)\})$$

for all Σ formulas F in which z does not occur free.

The precise relationship reads as follows:

5.1 Theorem. *$KP + \Sigma$ separation and $(\Pi_2^1 - CA) + BI$ prove the same sentences of second order arithmetic.*¹³

The ordinals κ such that $L_\kappa \models KP + \Sigma$ separation are familiar from ordinal recursion theory.

5.2 Definition. An admissible ordinal κ is said to be nonprojectible if there is no total κ -recursive function mapping κ one-one into some $\beta < \kappa$, where a function $F : L_\kappa \rightarrow L_\kappa$ is called κ -recursive if it is Σ definable in L_κ .

The key to the ‘largeness’ properties of nonprojectible ordinals is:

5.3 Theorem. *For any nonprojectible ordinal κ , L_κ is a limit of Σ_1 -elementary substructures¹⁴, i.e. for every $\beta < \kappa$ there exists a $\beta < \rho < \kappa$ such that L_ρ is a Σ_1 -elementary substructure of L_κ , written $L_\rho \prec_1 L_\kappa$.*

Such ordinals satisfying $L_\rho \prec_1 L_\kappa$ have strong reflecting properties. For instance, if $L_\rho \models F$ for some set-theoretic sentence F (possibly containing parameters from L_ρ), then there exists a $\gamma < \rho$ such that $L_\gamma \models F$. This is because $L_\rho \models F$ implies $L_\kappa \models \exists \gamma F^{L_\gamma}$, hence $L_\rho \models \exists \gamma F^{L_\gamma}$ using $L_\rho \prec_1 L_\kappa$.

The last result makes it clear that an ordinal analysis of Π_2^1 comprehension would necessarily involve a proof-theoretic treatment of reflections beyond those surfacing in admissible proof theory. Here one encounters two difficulties.

1. Significantly stronger notation systems are required. The problem is (as always in this area) to develop a constructive object, i.e. a notation system, that shares “enough” properties with a (recursively) large ordinal. So far definition procedures based on ideas of Veblen and Bachmann have been paramount, but it seems that this approach is constrained to admissible proof theory. So some new ideas will be needed.

¹³Warning: It is crucial to this result that Infinity is among the axioms of KP .

¹⁴ L_ρ is said to be a Σ_1 -elementary substructure of L_κ if every Σ_1 -sentence with parameters from L_ρ that holds in L_κ also holds in L_ρ .

2. New cut-elimination procedures have to be invented. Of course, this task cannot be completely separated from the previous one since the ideas giving rise to a notation system should lend themselves to a cut-elimination procedure.

Recently we have been able to get hold on Π_n -reflection for arbitrary n .

5.4 Definition. A set-theoretic formula is said to be Π_n (respectively Σ_n) if it consists of a string of n alternating quantifiers beginning with an universal one (respectively existential one), followed by a Δ_0 formula. By Π_n -reflection we mean the scheme

$$F \rightarrow \exists z[Tran(z) \wedge z \neq \emptyset \wedge F^z]$$

where F is Π_n , and $Tran(z)$ expresses that z is a transitive set.

$\alpha > 0$ is said to be Π_n -reflecting if $L_\alpha \models \Pi_n$ -reflection.

Σ_n -reflection and Σ_n -reflecting ordinals are defined analogously.

Π_n -reflecting ordinals have interesting points of contact with non-monotone inductive definitions.

5.5 Definition. A function Γ from the power set of $\mathbb{N}$ into itself is called operator on $\mathbb{N}$. Γ determines a transfinite sequence $\langle \Gamma^\xi : \xi \in ON \rangle$ of subsets of $\mathbb{N}$ defined by

$$\Gamma^\lambda = \Gamma^{<\lambda} \cup \Gamma(\Gamma^{<\lambda}),$$

where $\Gamma^{<\lambda} = \bigcup_{\xi < \lambda} \Gamma^\xi$.

The closure ordinal $|\Gamma|$ of Γ is the least ordinal ρ such that $\Gamma^{\rho+1} = \Gamma^\rho$.

Γ is said to be Π_k^0 in case there is an arithmetic Π_k^0 formula $F(U, u)$ with free second order variable U such that for $X \subseteq \mathbb{N}$,

$$\Gamma(X) = \{n \in \mathbb{N} : F(X, n)\}.$$

Let $|\Pi_k^0| := \sup\{|\Gamma| : \Gamma \text{ is } \Pi_k^0\}$.

By work of Aczel and Richter [1974] we have the following characterization.

5.6 Theorem.

$$|\Pi_k^0| = \text{first } \Pi_{k+1}\text{-reflecting ordinal}.$$

Several notions of recursively large ordinals are modelled upon notions of large cardinals. This is especially true of notions like “recursively inaccessible ordinal” and “recursively Mahlo ordinal”. It turns out that the least Π_3 -reflecting ordinal is greater than the least recursively Mahlo ordinal, indeed much greater than any iteration of “Mahloness” into the transfinite from below.

5.7 Definition. Assume that κ is recursively Mahlo. κ is called recursively α -Mahlo if for every κ -recursive function $f : \kappa \rightarrow \kappa$ there is an ordinal $\beta < \kappa$ closed under f such that β is recursively γ -Mahlo for any $\gamma < \alpha$.

κ is recursively hyper-Mahlo if κ is recursively κ -Mahlo.

As a matter of fact, there are ‘many’ recursively hyper–Mahlo ordinals below the first Π_3 –reflecting ordinal. Aczel and Richter [1974] have convincingly argued that Π_3 –reflecting ordinals are the recursive analogue of weakly compact cardinals also known as Π_1^1 –indescribable cardinals. The same considerations justify the view that Π_{n+2} –reflecting ordinals provide the recursive analogue for the Π_n^1 –indescribable cardinals for all $n > 0$.

Next we shall glimpse at an ordinal notation system which in some respect internalizes the first Π_3 –reflecting ordinal. Rather than exhibiting such a notation system, it is more appropriate to give a model for the peculiar functions the notation system is made from. Such a model can be provided on the basis of a weakly compact cardinal.

So let us indulge in a little science fiction and fix a weakly compact cardinal κ .

5.8 Definition. Let

$$V = \bigcup_{\alpha \in ON} V_\alpha$$

be the cumulative hierarchy of sets, i.e.

$$V_0 = \emptyset, \quad V_{\alpha+1} = \{X : X \subset V_\alpha\}, \quad V_\lambda = \bigcup_{\xi < \lambda} V_\xi \text{ for limit ordinals } \lambda.$$

A cardinal κ is *weakly compact* if whenever $U \subseteq V_\kappa$ and $A(P)$ is a Π_1^1 formula of set theory with P a class variable such that $\langle V_\kappa, \in \rangle \models A(U)$, then for some $\alpha < \kappa$:

$$\langle V_\alpha, \in \rangle \models A(U \cap V_\alpha).$$

For κ a regular cardinal, a subset $S \subseteq \kappa$ is *stationary in κ* if $S \cap C \neq \emptyset$ holds for every set $C \subseteq \kappa$ that is closed and unbounded in κ .

5.9 Definition. Let κ be a weakly compact cardinal. By recursion on α we define sets $B(\alpha, \beta)$, M^α and the function Ξ_κ as follows:

$$B(\alpha, \beta) = \begin{cases} \text{closure of } \beta \cup \{0, \kappa\} \\ \text{under } +, \lambda\xi \cdot \omega^\xi \text{ and } (\xi \mapsto \Xi_\kappa(\xi))_{\xi < \alpha} \end{cases}$$

$$M^\alpha = \{\pi < \kappa : B(\alpha, \pi) \cap \kappa = \pi \wedge \forall \xi \in B(\alpha, \pi) \cap \alpha [\pi \cap M^\xi \text{ stationary in } \pi]\}$$

$$\Xi_\kappa(\alpha) = \text{least element of } M^\alpha.$$

The hypothesis that κ be weakly compact will be needed to ensure that $M^\alpha \neq \emptyset$ and thus to show that $\Xi_\kappa(\alpha)$ is defined.

In a second step, for every $\pi \in M^\alpha$ and $\xi \in B(\alpha, \pi) \cap \alpha$, one defines collapsing functions

$$\Theta_\pi^\xi : ON \longrightarrow \pi \cap M^\xi.$$

With the aid of (symbols for) the functions and constants $\Xi_\kappa, \Theta_\pi^\xi, +, \omega, \kappa, 0$, and special constraints needed to ensure uniqueness of notations, it is then possible to construct a primitive recursive system of ordinal notations $N(\kappa)$ which reflects some properties of the rather large cardinal κ .

Akin to RS_Ω one can invent an infinitary calculus RS_κ , which in addition has the following rules:

$$(\Pi_3\text{-}Ref_\kappa) \frac{\Gamma, A^\kappa}{\Gamma, (\exists z \in \check{L}_\kappa)[Tran(z) \wedge z \neq \emptyset \wedge A^z]}$$

for every $\Pi_3(\kappa)$ formula A and

$$(\Pi_2\text{-}Ref_\pi^\xi) \frac{\Gamma, B}{\Gamma, (\exists z \in \check{L}_\pi)(z \in M^\xi \wedge B^z)}$$

for every $\Pi_2(\pi)$ -formula B , where $\pi \in M^\alpha$, $\xi < \alpha$, $\xi \in B(\alpha, \pi)$.

The rules $(\Pi_2\text{-}Ref_\pi^\xi)$ are not needed for an embedding of $KP + \Pi_3$ -reflection into RS_κ . They are only required for carrying through the cut-elimination procedure. Usually, removing one instance of $(\Pi_3\text{-}Ref_\kappa)$ in a derivation can be done only at the expense of introducing a bunch of new $(\Pi_2\text{-}Ref_\pi^\xi)$ rules. This discriminates the cut-elimination for RS_κ sharply from that for RS_Ω , where instances of the impredicative rule $(\Sigma\text{-}Ref_\Omega)$ are replaced by instances of the predicative rule $(\exists)$.

Cut-elimination for RS_κ can be achieved by using the $\mathcal{H}$ -controlled RS_κ -derivations, with $\mathcal{H}$ ranging over the operators

$$\mathcal{H}_\gamma(X) = \bigcap \{B(\alpha, \beta) : X \subseteq B(\alpha, \beta) \wedge \gamma < \alpha \wedge \beta < \kappa\}$$

where $\gamma \in N(\kappa)$.

For $\Gamma = \{A_1, \dots, A_n\}$ we set $\Gamma^\pi := \{A_1^\pi, \dots, A_n^\pi\}$.

The key to the elimination of $(\Pi_3\text{-}Ref_\kappa)$ is the following theorem.

5.10 Theorem. *If Γ is a set of $\Pi_3(\kappa)$ formulas and $\mathcal{H}_\gamma \upharpoonright_{\frac{\alpha}{\kappa+1}} \Gamma$, then, for every $\pi \in M^{f(\alpha, \gamma)}$,*

$$\mathcal{H}_{f(\alpha, \gamma)}[\pi] \upharpoonright_{\frac{\Xi_\kappa(f(\alpha, \gamma) + \pi)}{\Xi_\kappa(f(\alpha, \gamma) + \pi)}} \Gamma^\pi,$$

where f is a function that depends only on Γ .

It is not by accident that in Theorem 5.9 a single derivation is 'collapsed' into a family of derivations indexed by a stationary subset of κ . The elimination of $(\Pi_3\text{-}Ref_\kappa)$ requires such a "stationary collapsing" technique.

Unfortunately, we will not be able to go any further into details. The interested reader is referred to Rathjen [1991e].

At the end we hasten to assure that this is not the first of an infinite series of new cut-elimination procedures. Π_3 -reflection just served as a paradigm. Stationary collapsing is applicable to all of the theories $KP + \Pi_n$ -reflection.

To close, we raise the question of how far afield from Π_2^1 comprehension all this is. The idea is to approach Π_2^1 comprehension by stronger and yet stronger reflection principles in an autonomous manner. I conjecture that the large cardinal analogue for a suitable notation system resides below the first Ramsey cardinal, and, moreover, is compatible

with $V = L$.

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